Parafermionic observables in planar statistical physics models

Hugo Duminil-Copin, Université de Genève

2013
I. Warming up!

II. Phase diagram of the FK percolation (non rigorous)

III. Rigorous results (without parafermionic observable)

IV. Rigorous results (with parafermionic observable)
Self-avoiding walk

On the hexagonal lattice, consider self-avoiding walks of length $n$ starting at the origin [Flory, Ott, '50s].
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Proposition [Hammersley]

$\left( \# \text{SAWs of length } n \right)^{1/n} \rightarrow \mu_c$ (called the connective constant).
Theorem [D-C, Smirnov, 2010]

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F(z) := \sum_{\gamma \in \mathcal{D}: \, \alpha \to z} \mu^{-|\gamma|}.
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**Definition**

The **winding** $W_\Gamma(a, b)$ of a curve $\Gamma$ between $a$ and $b$ is the rotation (in radians) of the curve between $a$ and $b$.

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**Definition**

The **winding** $W_\Gamma(a, b)$ of a curve $\Gamma$ between $a$ and $b$ is the rotation (in radians) of the curve between $a$ and $b$.

The **parafermionic operator** at a mid-point $z \in \mathcal{D}$ is defined by

$$F(z) := \sum_{\gamma \subset \mathcal{D}: a \rightarrow z} e^{-i\sigma W_\gamma(a, z)} \mu^{-|\gamma|}.$$
Lemma (Local relation around a vertex)

If $\sigma = \frac{5}{8}$ and $\mu = \sqrt{2} + \sqrt{2}$, then $F$ satisfies the following relation for every vertex $v \in V(D)$,

$$(p - v)F(p) + (q - v)F(q) + (r - v)F(r) = 0$$

where $p, q, r$ are the mid-edges of the three edges adjacent to $v$. 

This relation means that $\oint_C F(z) \, dz = 0$ along the contour $qvrp$.
Lemma (Local relation around a vertex)

If \( \sigma = \frac{5}{8} \) and \( \mu = \sqrt{2 + \sqrt{2}} \), then \( F \) satisfies the following relation for every vertex \( v \in V(D) \),

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Lemma (Local relation around a vertex)

If $\sigma = \frac{5}{6}$ and $\mu = \sqrt{2} + \sqrt{2}$, then $F$ satisfies the following relation for every vertex $v \in V(D)$,

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This relation means that $\oint F(z)dz = 0$ along the contour.

Proposition (Discrete holomorphicity)

If $D$ is simply connected, then $\int_C F(z)dz = 0$ for any discrete contour $C$. 
Lemma (Local relation around a vertex)

If $\sigma = \frac{5}{3}$ and $\mu = \sqrt{2 + \sqrt{2}}$, then $F$ satisfies the following relation for every vertex $v \in V(D)$,

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Proposition (Discrete holomorphicity)

If $D$ is simply connected, then $\oint_C F(z)dz = 0$ for any discrete contour $C$.

These relations do not determine the observable from its boundary conditions.
If we consider the exterior boundary of this trapeze, we obtain

When $\sigma = 5$ and $\mu = \sqrt{2 + \sqrt{2}}$,

$$0 = -\sum_{z \in \text{bottom}} F(z) + \sum_{z \in \text{top}} F(z) + e^{i 2 \pi/3} \sum_{z \in \text{left}} F(z) + e^{-i 2 \pi/3} \sum_{z \in \text{right}} F(z).$$

The winding on the boundary is deterministic! Thus, $F$ can be replaced by the sum of Boltzmann weights.

Last ingredient. The result follows from combinatorial arguments.
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When $\sigma = \frac{5}{8}$ and $\mu = \sqrt{2 + \sqrt{2}}$, we find

$$1 = \cos\left(\frac{3\pi}{8}\right) \sum_{\gamma: a \to \text{bottom}} \mu^{-|\gamma|} + \sum_{\gamma: a \to \text{top}} \mu^{-|\gamma|} + \cos\left(\frac{\pi}{4}\right) \sum_{\gamma: a \to \text{sides}} \mu^{-|\gamma|}.$$

💡 The winding on the boundary is deterministic! Thus, $F$ can be replaced by the sum of Boltzman weights.
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**Last ingredient.** The result follows from combinatorial arguments.
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**Potts model** [Potts, Domb, 1951]

Consider $q$ colors. Assign to each site $x$ outside $B_n = [-n, n]^2$ the color $\sigma_x = 1$ and each site $x \in B_n$ an arbitrary color $\sigma_x \in \{1, \ldots, q\}$ according to the following probability measure:

$$\mathbb{P}^{(1)}_{T, q, n}[\sigma] \propto \exp(-H(\sigma)/T) \quad \text{where} \quad H(\sigma) := \text{card}(x \sim y \text{ with } \sigma_x \neq \sigma_y).$$
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This model undergoes a phase transition in infinite volume at critical temperature $T_c(q)$:

$$\lim_{n \to \infty} \mathbb{P}^{(1)}_{T, q, n}[\sigma_0 = 1] = \begin{cases} 
1 & \text{if } T > T_c(q), 
\frac{1}{q} + \frac{m(T)}{q} & \text{if } T < T_c(q).
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A geometrical representation of Potts models: the FK percolation

This percolation model [Fortuin-Kasteleyn, 1969] is defined as follows. Edges outside $B_n$ are open. Each edge in $B_n$ is open or closed. The probability of a configuration $\omega \in \{\text{open, closed}\}^{E(B_n)}$ is given by the formula

$$
\phi_{p,q,n}^w(\omega) := \frac{1}{Z_{p,q,n}} \cdot p^\#\text{open edges} \cdot (1 - p)^\#\text{closed edges} \cdot q^\#\text{connected components}.
$$

For $q = 1$, the model is Bernoulli percolation. For $q \geq 1$, in infinite volume, there exists $p_c(q) \in (0, 1)$ such that

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\phi_{p,q,n}(\omega) = \begin{cases} 
0 & \text{if } p < p_c(q), \\
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\phi_{p,q,\mathbb{Z}^2}^w(0 \leftrightarrow \infty) = \begin{cases} 
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The \( q \)-states Potts model can be obtained from the FK percolation with cluster weight \( q \in \mathbb{N} \setminus \{0, 1\} \) by coloring each cluster independently.
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This coupling provides us with a dictionary between properties of FK percolation and Potts models. For instance, the transition exists and the critical point follows by considerations of duality $T_c(q) = -1/\log(1 - p_c(q))$. 
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$$\mathbb{P}_{T(p),q,n}[\sigma_0 = 1] = \frac{1}{q} + \left(1 - \frac{1}{q}\right) \phi_{p,q,n}^{w}(0 \leftrightarrow \partial B_n)$$
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\mathbb{P}_{T(p),q,n}^{(1)}[\sigma_0 = 1] = \frac{1}{q} + \left(1 - \frac{1}{q}\right) \phi_{p,q,n}^w(0 \leftrightarrow \partial B_n)
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\[T_c(q) = -1/\log(1 - p_c(q)).\]
A first guess for $p_c(q)$

A **dual model** can be defined on the dual lattice $(\mathbb{Z}^2)^* = (\frac{1}{2}, \frac{1}{2}) + \mathbb{Z}^2$:
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- For FK percolation, the dual model is a FK percolation with $p^*$ and $q^*$ defined by

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q^* = q \quad \text{and} \quad \frac{pp^*}{(1 - p)(1 - p^*)} = q.
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**Conjecture** [Potts, 1952] $p_c(q) = p_c(q)^* = \frac{\sqrt{q}}{1 + \sqrt{q}}$. 

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$p_c(q) = \frac{\sqrt{q}}{1+\sqrt{q}}$ [Potts 52]

Discontinuous [Baxter 73]

Continuous [Baxter 73]
Parafermionic observables in planar statistical physics models

\[ p_c(q) = \frac{\sqrt{q}}{1+\sqrt{q}} \]

- **Subcritical phase (exponential decay)**
  
  \( q \geq 25, \text{72 regime Pirogov-Sinai} \)
  
  [Kotecký-Schlosman, 80s]

- **Supercritical phase (infinite cluster)**

  Discontinuous [Baxter 73]

  Critical line for \( q \geq 4 \) [Hinterman et al, 78]

- **FK Ising**

  Continuous [Baxter 73]

  [Onsager, Wu 50s]

  [Kesten 80]
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II. Phase diagram of the FK percolation (non rigorous)

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IV. Rigorous results (with parafermionic observable)
Theorem [Beffara, D-C, 2010]

Let $q \geq 1$ and $p < p_c(q)$, there exists $\tau = \tau(p, q) > 0$ such that

$$\phi_{p,q,\mathbb{Z}^2}^w(0 \leftrightarrow x) \leq e^{-\tau|x|} \quad \text{for any } x \in \mathbb{Z}^2.$$ 

The proof is based on

- Considerations of both the model and its dual but no discrete holomorphicity.

- A sharp threshold argument for boolean functions coming from combinatorics.
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Let $q \geq 1$ and $p < p_c(q)$, there exists $\tau = \tau(p, q) > 0$ such that

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Corollaries

- (For FK) $p_c(q) = \frac{\sqrt{q}}{1+\sqrt{q}}$ when $q \geq 1$. 

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- (For FK) $p_c(q) = \frac{\sqrt{q}}{1+\sqrt{q}}$ when $q \geq 1$.
- (For Potts) $T_c(q) = \frac{1}{\ln(1+\sqrt{q})}$ for $q \geq 2$. 
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- (For Potts) $T_c(q) = \frac{1}{\ln(1 + \sqrt{q})}$ for $q \geq 2$.
- (For FK and Potts) Turn several results on subcritical and supercritical regimes into unconditional results.
Theorem [D-C, Sidoravicius, Tassion, 2017]

Consider the FK percolation with parameters $q \geq 1$ and $p = p_c(q)$. The following properties are equivalent:

1. Absence of infinite cluster at criticality: $\phi_{w_{p_c(q)},Z_2}(0\leftarrow\infty) = 0$,
2. Uniqueness of infinite measures: $\phi_f_{p_c(q),Z_2} = \phi_f_{p_c(q),Z_2}$,
3. Infinite susceptibility at criticality: $\phi_f_{p_c(q),Z_2}(\vert C \vert) := \sum_{x \in Z_2} \phi_f_{p_c(q),Z_2}(0\leftarrow x) = \infty$,
4. Absence of exponential decay of correlations: $\lim_{\vert x \vert \to \infty} \frac{1}{\vert x \vert \log \phi_{f_{p_c(q),Z_2}(0\leftarrow x)}} = 0$,
5. Strong form of RSW: There exists $c > 0$ such that for any rectangle $R_n = [0, 2^n] \times [0, n]$, $\phi_{f_{p_c(q),B_3}(R_n)\text{is crossed from left to right)} > c$. 

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4. Absence of exponential decay of correlations:

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\lim_{|x| \to \infty} \frac{1}{|x| \log \phi_{p_c,q,\mathbb{Z}^2}^f(0 \leftrightarrow x)} = 0,
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The loop representation of the FK percolation (dense Temperley-Lieb model)

- Consider both the primal and the dual models at the critical point
  \[ p = \sqrt{q}/(1 + \sqrt{q}) \]:

![Diagram showing the loop representation of the FK percolation]
The loop representation of the FK percolation (dense Temperley-Lieb model)

Consider both the primal and the dual models at the critical point $p = \sqrt{q} / (1 + \sqrt{q})$:
The loop representation of the FK percolation (dense Temperley-Lieb model)

- Consider both the primal **and** the dual models at the critical point $p = \sqrt{q}/(1 + \sqrt{q})$:

- It is a Temperley-Lieb loop model: the **probability of a configuration** is given by:

$$\phi_{p_c, q, D}(\omega) = \frac{\sqrt{q}^{\text{#loops}}}{Z(D, q)}$$
Let $\mathcal{D}$ be a **discrete domain** with two prescribed points $a$ and $b$ on the boundary.

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The loop representation of this model is a **collection of loops** and a **single curve** from $a$ to $b$ called the **exploration path** $\gamma$. 
For any mid-edge of the medial lattice, the so-called parafermionic observable $F$ is defined as:

$$F(e) = \mathbb{E}_{\rho_c, q, \mathcal{D}}^{a, b} \left[ e^{i\sigma W(e, b)} \mathbb{1}_{e \in \gamma} \right].$$

Define the spin $\sigma$ satisfying $\sin(\sigma \pi/2) = \sqrt{q^2}$. This observable satisfies a local relation:

$$F(e_1) - F(e_3) = i \left[ F(e_2) - F(e_4) \right].$$

These relations do not determine $F$, but one can integrate along discrete contours to obtain relevant information.
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Theorem [D-C, 2012] [D-C, Sidoravicius, Tassion, 2013]

For $1 \leq q \leq 4$, the transition is continuous at $p_c$. 

Exploit the fact that discrete contour integrals vanish on universal cover of $\mathbb{Z}^2$ minus a face.

Some delicate probability arguments to go from the universal cover geometry to $\mathbb{Z}^2$.

Corollary

- No spontaneous magnetization for critical 2, 3 and 4 Potts models
- Existence of polynomial bounds for arm-exponents.
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Conjecture [Schramm, 2006]

Let $q \in (0, 4]$ and consider a domain $(\mathcal{D}, a, b)$. The exploration path $\gamma_\delta$ converges in law (as $\delta \to 0$) to an SLE($\kappa$) where

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\[
\Phi \begin{array}{c}
\gamma \\
\Phi(\gamma)
\end{array}
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\Phi \begin{array}{c}
a \\
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FK representation of the Ising model (cluster weight $q = 2$): The spin equals $\sigma = \frac{1}{2}$, thus determining the complex argument of the observable. Stanislav Smirnov used this fact to prove the convergence of the (para)fermionic observable.

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Parafermionic observables in planar statistical physics models

$p_c(q) = \frac{\sqrt{q}}{1+\sqrt{q}}$

- subcritical phase (exponential decay)
- supercritical phase (infinite cluster)

$\text{SLE} \left( \frac{4\pi}{\arccos(-\sqrt{q}/2)} \right)$ [Schramm, 06]

- Discontinuous [Baxter 73]
- Continuous SLE (16/3)

- FK Ising
- UST
- percolation
- $p\rightarrow 1$
Open questions

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- Do the same with **loop $O(n)$-models**:
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- Do the same with **loop \( O(n) \)-models**:
  In particular prove Nienhuis’s conjecture that \( x_c(n) = \frac{1}{\sqrt{2+\sqrt{2-n}}} \) for \( n \in (-2, 2) \).
Thank you