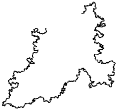
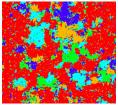
# Parafermionic observables in planar statistical physics models

Hugo Duminil-Copin, Université de Genève

2013



**I.** Warming up!



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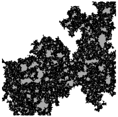

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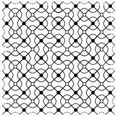
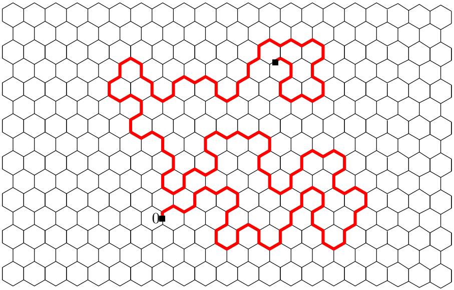
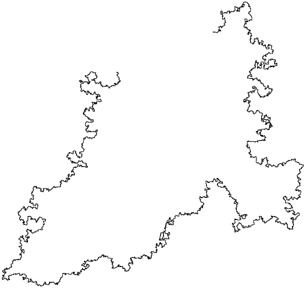


**IV.** Rigorous results (with parafermionic observable)

# Self-avoiding walk

On the hexagonal lattice, consider self-avoiding walks of length $n$ starting at the origin [Flory, Ott, '50s].



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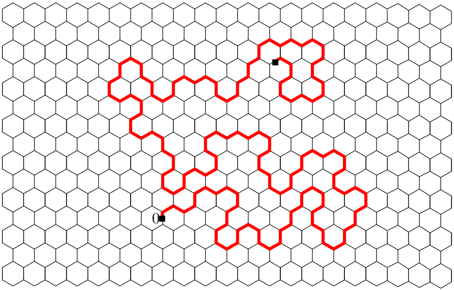


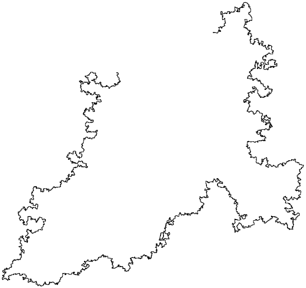

**Proposition** [Hammersley]

$(\#\text{ SAWs of length } n)^{1/n} \longrightarrow \mu_c$ (called the connective constant).

**Theorem** [D-C, Smirnov, 2010]

The connective constant of the **hexagonal lattice** is equal to $\sqrt{2+\sqrt{2}}$.

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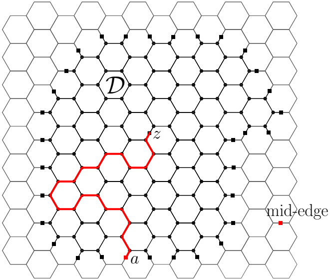


$$F(z) := \sum_{\gamma \subset \mathcal{D}:\ a \to z} \mu^{-|\gamma|}.$$

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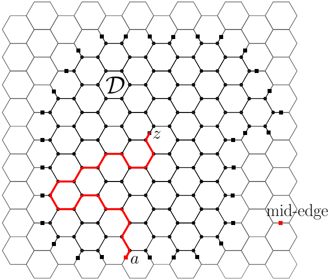


## Definition

The winding $\mathrm{W}_\Gamma(a,b)$ of a curve $\Gamma$ between $a$ and $b$ is the rotation (in radians) of the curve between $a$ and $b$.

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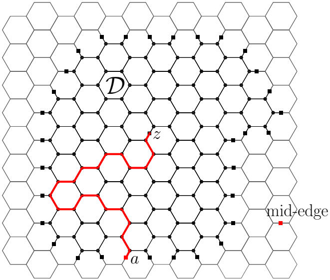


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The **parafermionic operator** at a mid-point $z \in \mathcal{D}$ is defined by

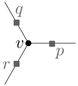
$$F(z) := \sum_{\gamma \subset \mathcal{D}:\ a \to z} \mathrm{e}^{-\mathrm{i}\sigma \mathrm{W}_\gamma(a,z)} \mu^{-|\gamma|}.$$

## Lemma (Local relation around a vertex)

*If $\sigma = \frac{5}{8}$ and $\mu = \sqrt{2+\sqrt{2}}$, then $F$ satisfies the following relation for every vertex $v \in V(\mathcal{D})$,*

$$(p-v)F(p) + (q-v)F(q) + (r-v)F(r) = 0$$

*where $p, q, r$ are the mid-edges of the three edges adjacent to $v$.*



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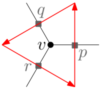

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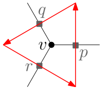


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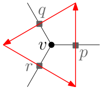


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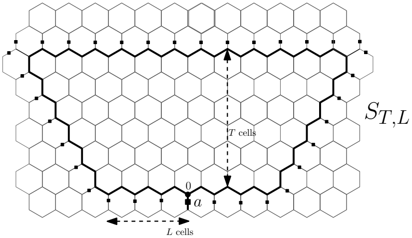
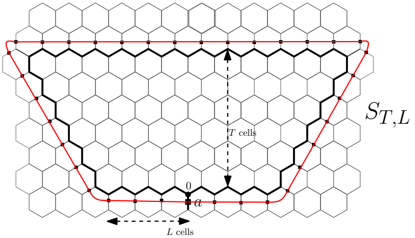
These relations do not determine the observable from its boundary conditions.



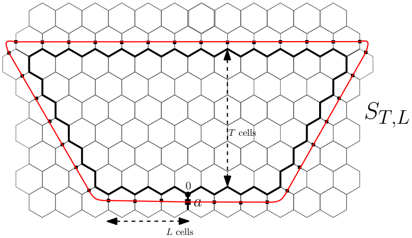
$S_{T,L}$
$T$ cells
0
$a$
$L$ cells




If we consider the exterior boundary of this trapeze, we obtain

When $\sigma = \frac{5}{8}$ and $\mu = \sqrt{2+\sqrt{2}}$,

$$0 = -\sum_{z\in \textit{bottom}} F(z) + \sum_{z\in \textit{top}} F(z) + \mathrm{e}^{\mathrm{i}\frac{2\pi}{3}} \sum_{z\in \textit{left}} F(z) + \mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}} \sum_{z\in \textit{right}} F(z)$$

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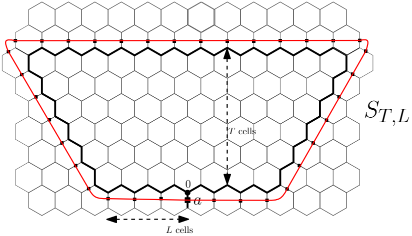
The winding on the boundary is deterministic! Thus, $F$ can be replaced by the sum of Boltzman weights.

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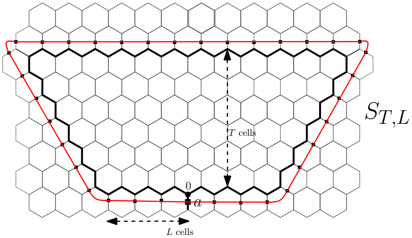


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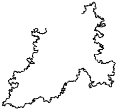
Last ingredient. The result follows from combinatorial arguments.



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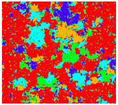

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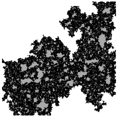

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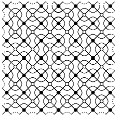

**IV.** Rigorous results (with parafermionic observable)

## Potts model [Potts, Domb, 1951]

Consider $q$ colors. Assign to each site $x$ outside $B_n = [-n, n]^2$ the color $\sigma_x = 1$ and each site $x \in B_n$ an arbitrary color $\sigma_x \in \{1, \ldots, q\}$ according to the following probability measure:

$$\mathbb{P}^{(1)}_{T,q,n}[\sigma] \propto \exp(-H(\sigma)/T) \quad \text{where} \quad H(\sigma) := \text{card}\big(x \sim y \text{ with } \sigma_x \neq \sigma_y\big).$$

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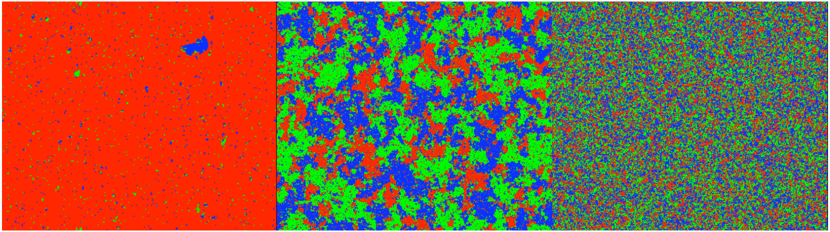

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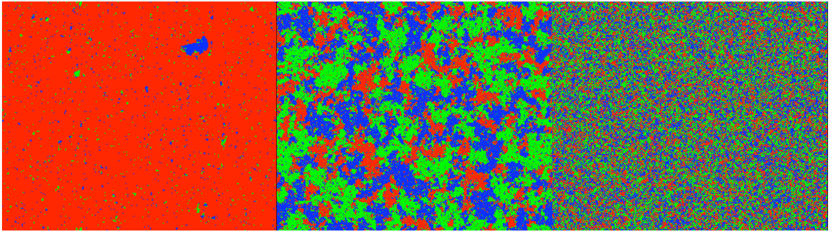

This model undergoes a phase transition in infinite volume at critical temperature $T_c(q)$:

$$\lim_{n\to\infty} \mathbb{P}^{(1)}_{T,q,n}[\sigma_0 = 1] = \begin{cases} \frac{1}{q} & \text{if } T > T_c(q), \\ \frac{1}{q} + m(T) > \frac{1}{q} & \text{if } T < T_c(q). \end{cases}$$

## A geometrical representation of Potts models: the FK percolation

This **percolation model** [Fortuin-Kasteleyn, 1969] is defined as follows. Edges outside $B_n$ are open. Each edge in $B_n$ is open or closed. The probability of a configuration $\omega \in \{open, closed\}^{E(B_n)}$ is given by the formula

$$\phi^{\mathrm{w}}_{p,q,n}(\omega) \;:=\; \frac{1}{Z_{p,q,n}} \cdot p^{\#\text{open edges}}(1-p)^{\#\text{closed edges}} q^{\#\text{connected components}}.$$

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- For $q \geq 1$, in infinite volume, there exists $p_c(q) \in (0,1)$ such that

$$\phi^{\mathrm{w}}_{p,q,\mathbb{Z}^2}(0 \leftrightarrow \infty) = \begin{cases} 0 & \text{if } p < p_c(q), \\ \theta_q(p) > 0 & \text{if } p > p_c(q). \end{cases}$$

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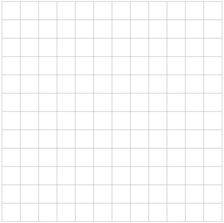
The $q$-states Potts model can be obtained from the FK percolation with cluster weight $q \in \mathbb{N} \setminus \{0, 1\}$ by coloring each cluster independently.



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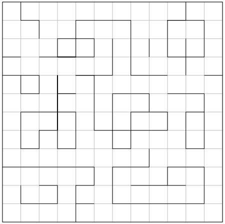

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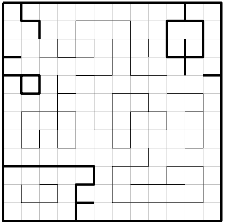

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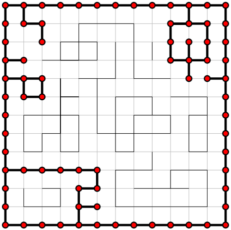

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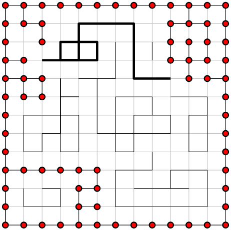

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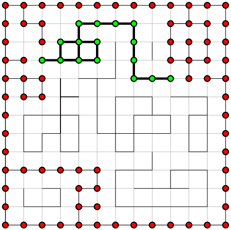

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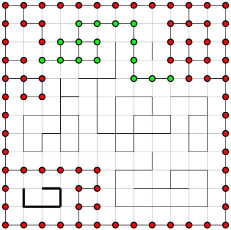

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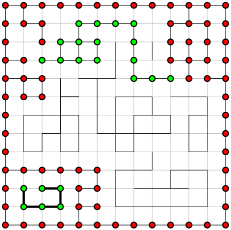

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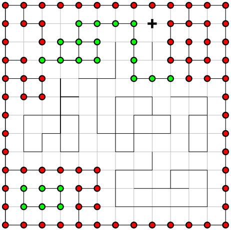

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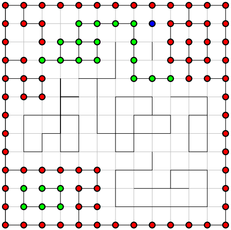

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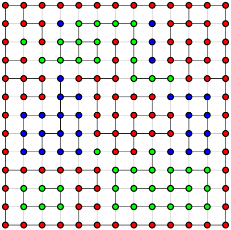

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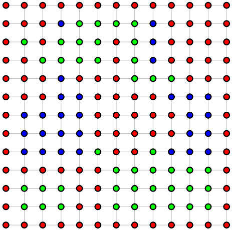

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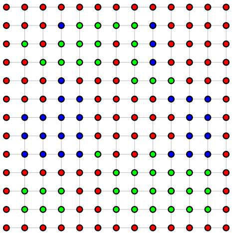

This coupling provides us with a **dictionary** between properties of FK percolation and Potts models. For instance,

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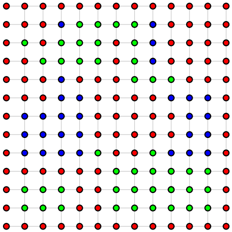
## A geometrical representation of Potts models: the FK percolation (2)

The $q$-states Potts model can be obtained from the FK percolation with cluster weight $q \in \mathbb{N} \setminus \{0, 1\}$ by coloring each cluster independently.



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The transition exists and the critical point follows by considerations of duality $\quad T_c(q) = -1/\log(1 - p_c(q))$.

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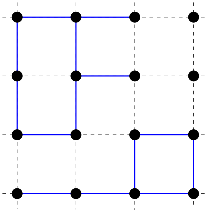
A **dual model** can be defined on the dual lattice $(\mathbb{Z}^2)^* = (\frac{1}{2}, \frac{1}{2}) + \mathbb{Z}^2$:



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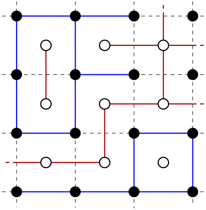

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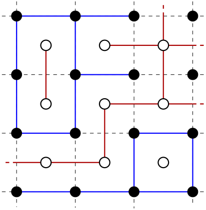

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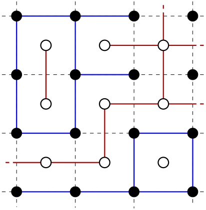

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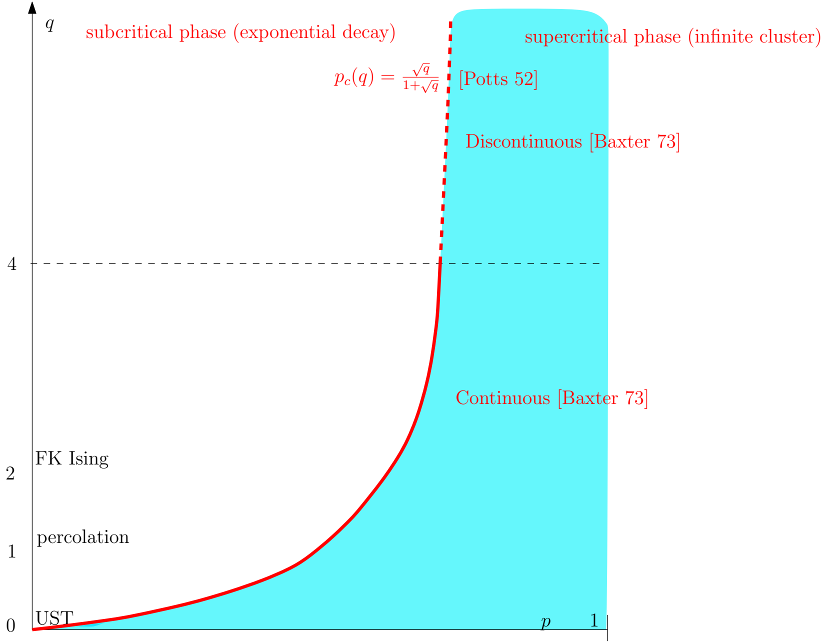
**Conjecture** [Potts, 1952] $\quad p_c(q) = p_c(q)^* = \frac{\sqrt{q}}{1+\sqrt{q}}$.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subcritical phase (exponential decay)
supercritical phase (infinite cluster)
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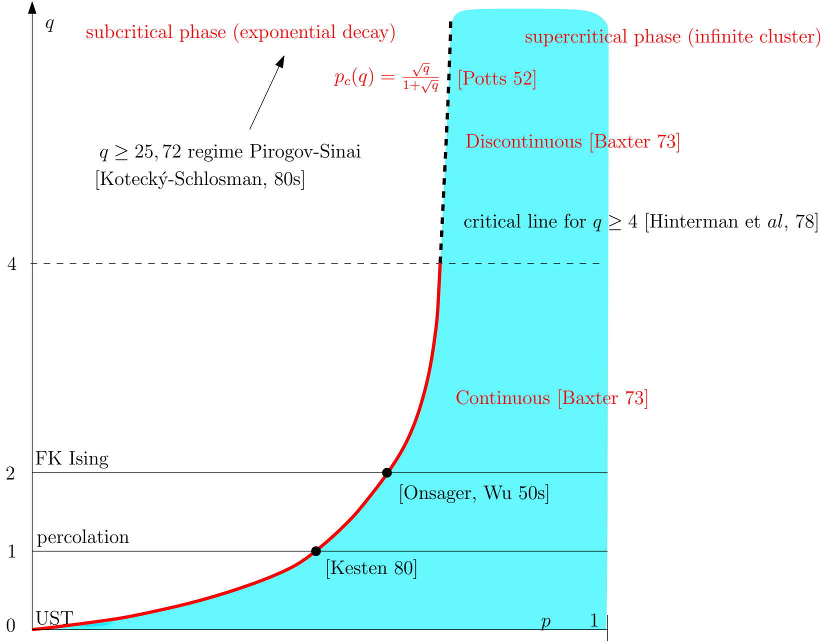

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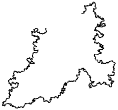

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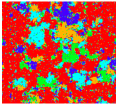

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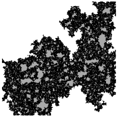

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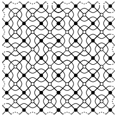

**IV.** Rigorous results (with parafermionic observable)

**Theorem** [Beffara, D-C, 2010]

Let $q \geq 1$ and $p < p_c(q)$, there exists $\tau = \tau(p,q) > 0$ such that

$$\phi^{\mathrm{w}}_{p,q,\mathbb{Z}^2}(0 \longleftrightarrow x) \leq e^{-\tau|x|} \qquad \text{for any } x \in \mathbb{Z}^2.$$

The proof is based on

- Considerations of both the model and its dual but no discrete holomorphicity.
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- (For FK and Potts) Turn several results on subcritical and supercritical regimes into unconditional results.

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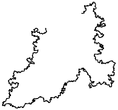
(5) Strong form of RSW: There exists $c > 0$ such that for any rectangle $R_n = [0, 2n] \times [0, n]$,

$$\phi^{f}_{p_c,q,B_{3n}}(R_n \text{ is crossed from left to right}) > c.$$



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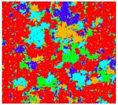

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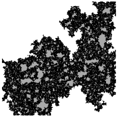

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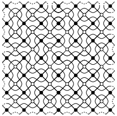

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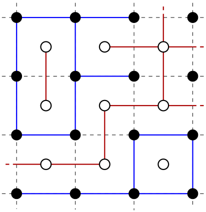
## The loop representation of the FK percolation (dense Temperley-Lieb model)

- Consider both the primal **and** the dual models at the critical point $p = \sqrt{q}/(1+\sqrt{q})$:



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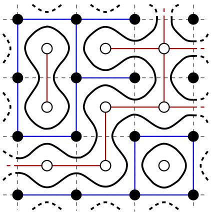

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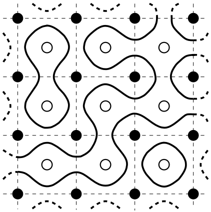
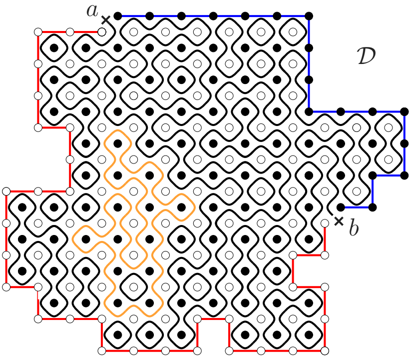


- It is a Temperley-Lieb loop model: the **probability of a configuration** is given by:

$$\phi_{p_c,q,\mathcal{D}}(\omega) = \frac{\sqrt{q}^{\#\text{loops}}}{Z(\mathcal{D},q)}$$

- Let $\mathcal{D}$ be a **discrete domain** with two prescribed points $a$ and $b$ on the boundary.

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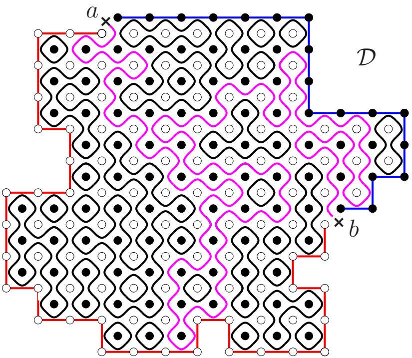


- We consider the loop model with **Dobrushin** boundary conditions.
- The loop representation of this model is a **collection of loops** and **a single curve** from $a$ to $b$ called the **exploration path** $\gamma$.

For any **mid-edge of the medial lattice**, the so-called **parafermionic observable** $F$ is defined as:

$$F(e) = \mathbb{E}^{a,b}_{p_c,q,\mathcal{D}}\left[\mathrm{e}^{\mathrm{i}\sigma W(e,b)}\mathbb{1}_{e\in\gamma}\right].$$

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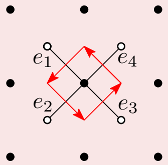

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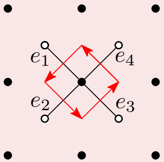


These relations **do not** determine $F$, but one can integrate along discrete contours to obtain relevant information.

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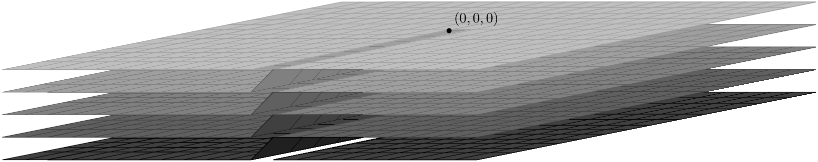

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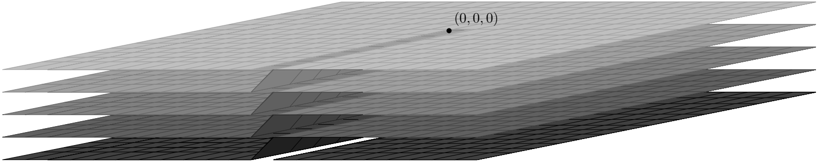


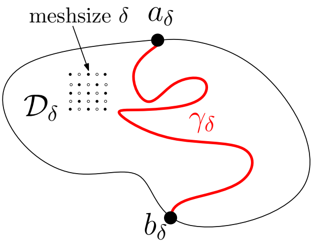

## Corollary

- No spontaneous magnetization for critical 2, 3 and 4 Potts models
- Existence of polynomial bounds for arm-exponents.
- Computation of universal arm-exponents for $1 \leq q < 4$.
- Spatial mixing properties.
- Sub-sequential limits of exploration paths can be parametrized by Loewner chains.

## Next step: Connections with conformal invariance

**Conjecture** [Schramm, 2006]

Let $q \in (0,4]$ and consider a domain $(\mathcal{D}, a, b)$. The exploration path $\gamma_\delta$ converges in law (as $\delta \to 0$) to an SLE($\kappa$) where $\kappa = \kappa(q) = \frac{4\pi}{\arccos(-\sqrt{q}/2)}$.



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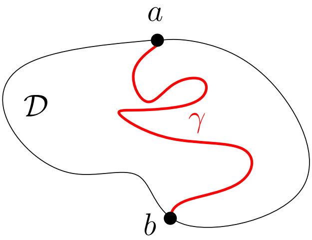

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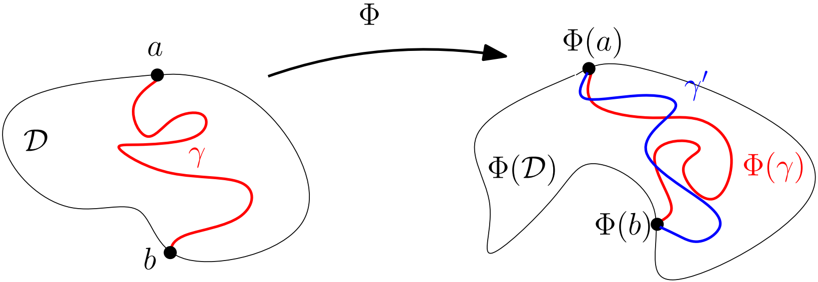

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*FK representation of the Ising model (cluster weight $q = 2$):*

The spin equals $\sigma = \frac{1}{2}$, thus determining the complex argument of the observable. Stanislav Smirnov used this fact to prove the convergence of the (para)fermionic observable.

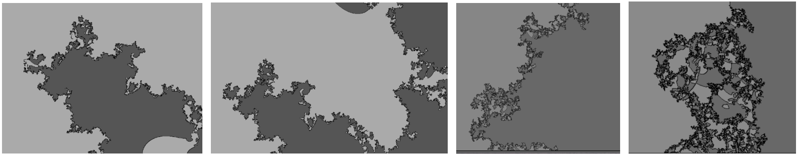
**Theorem** [Chelkak, D-C, Hongler, Kemppainen, Smirnov, 2012]

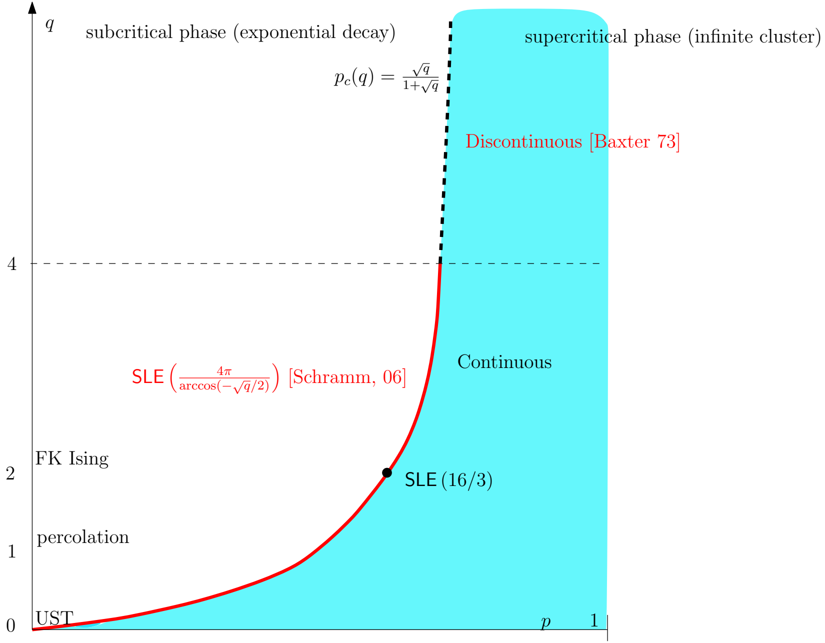
The exploration path converges to SLE$(16/3)$ for $q = 2$.

# Next step: Connections with conformal invariance

**Conjecture** [Schramm, 2006]

Let $q \in (0,4]$ and consider a domain $(\mathcal{D}, a, b)$. The exploration path $\gamma_\delta$ converges in law (as $\delta \to 0$) to an SLE($\kappa$) where $\kappa = \kappa(q) = \frac{4\pi}{\arccos(-\sqrt{q}/2)}$.





$q$
subcritical phase (exponential decay)
supercritical phase (infinite cluster)
$p_c(q) = \frac{\sqrt{q}}{1+\sqrt{q}}$
Discontinuous [Baxter 73]
4
Continuous
SLE $\left(\frac{4\pi}{\arccos(-\sqrt{q}/2)}\right)$ [Schramm, 06]
FK Ising
2
SLE (16/3)
percolation
1
UST
0
$p$
1

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- Do the same with loop $O(n)$-models:

  In particular prove Nienhuis's conjecture that $x_c(n) = \frac{1}{\sqrt{2+\sqrt{2-n}}}$ for $n \in (-2, 2)$.

# Thank you