In memory of Walter Hunziker (1935 - 2012)

Analytical Aspects of Mathematical Physics
May 29, 2013, ETH Zürich

A few milestones in Walter's life

1935	Born on the 9th of November
1954	Student of physics at ETH
1961	PhD under the advice of Res Jost
1963-65	Time in Princeton
1965/69/72	Assistant/Associate/Full Professor at ETH
2001	Retirement
2012	Died on the 9th of September

Students

Alex Schtalheim (1970)
Rolf Bodmer (1972)
Charles-Edouard Pfister (1974)
Egon Vock (1980)
Michael Loss (1982)
Gian Michele Graf (1990)
Volker Bach (1992)
Marcel Griesemer (1996)
Laura Cattaneo (2003)

Research interests

Mainly quantum mechanics, but also statistical mechanics

- Scattering theory (quantum and classical)
- Singular perturbation theory
- Resonances

Hunziker's Theorem (aka HVZ)

On the continuous spectrum of many-body Hamiltonians (in center of mass frame)

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- Hamiltonian of N particles

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where $a = (C_1, \ldots C_k)$ is a cluster decomposition.

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E.g. $N = 5$, $a = (13)(24)(5)$.

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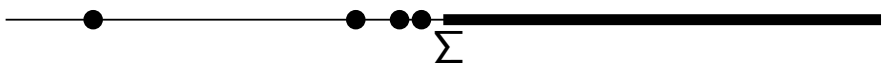

Spectrum of H

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Theorem For pair potentials decaying in $x_i - x_j$, the essential spectrum of H is

$$\begin{aligned} \sigma_{\text{ess}}(H) &= \bigcup_{a \neq (1 \ldots N)} \sigma(H_a) \\ &= [\Sigma, +\infty) \qquad (\Sigma := \inf \text{R.H.S.}) \end{aligned}$$

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- Proof of $\subset$ for $N = 2$: Resolvents $G(z) = (z - H)^{-1}$, $G_0(z) = (z - H_0)^{-1}$

$$G(z) = G_0(z) + G_0(z) V G(z) \quad \text{i.e.} \quad (1 - G_0(z) V) G(z) = G_0(z)$$

- $G_0(z) V$ compact ($z \notin \sigma(H_0)$), $\to 0$, ($\operatorname{Re} z \to -\infty$)

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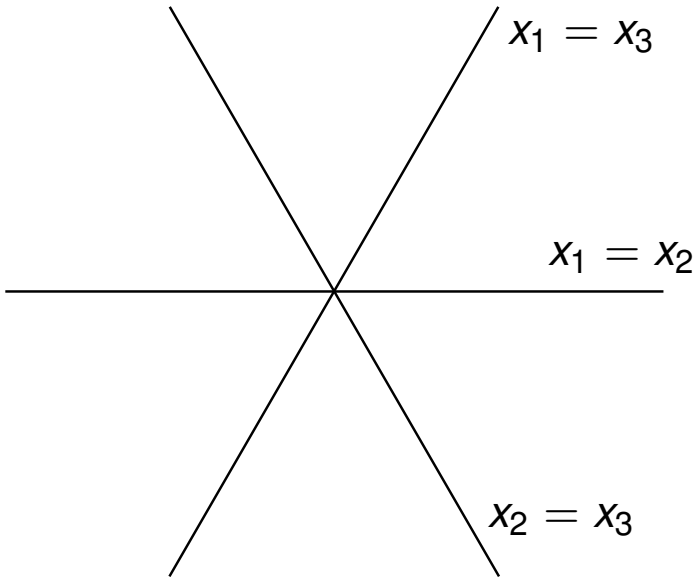
  - Hence $G(z)$ defined in $z \notin \sigma(H_0) \cup \{\text{poles}\}$, i.e., $\sigma_{\text{ess}}(H) \subset \sigma(H_0)$.

The general case: arbitrary N

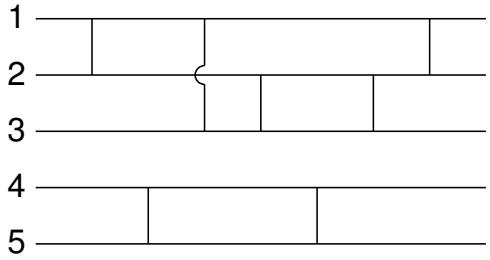
- $G_0(z)V$ is not compact for $V = \sum_{i<j} V_{ij}$, $(N \geq 3)$

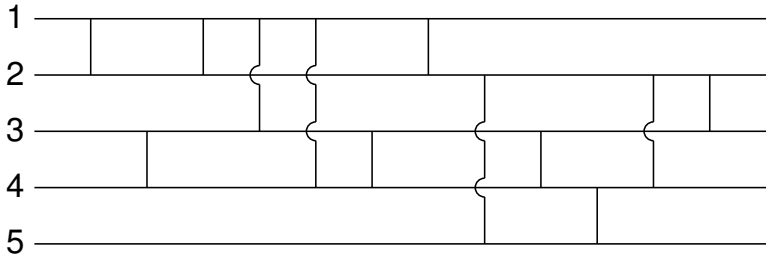


Configuration space for $N = 3$

The general case: arbitrary N

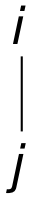
- $G_0(z)V$ is not compact for $V = \sum_{i<j} V_{ij}$, $(N \geq 3)$
- Iteration of $G(z) = G_0(z) + G_0(z)VG(z)$ yields diagrams $(N = 5)$





with

$G_0(z)$

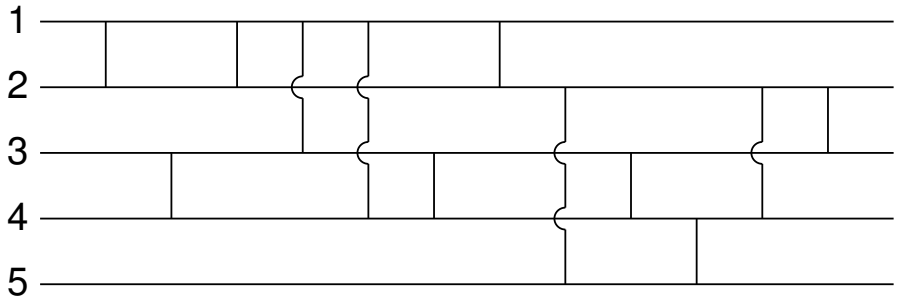


V_{ij}

- $$G(z) = D(z) + C(z)$$

  (sums of disconnected/connected diagrams; convergent for large $\mathrm{Re}\, z < 0$)

The Weinberg-van Winter equation



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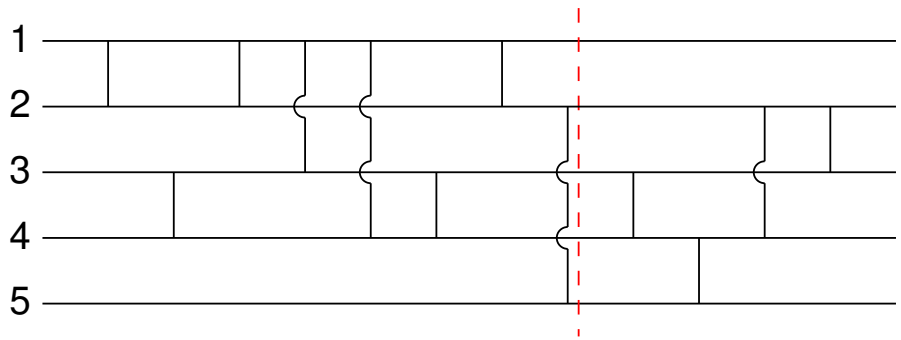

$$G(z) = D(z) + C(z) \qquad C(z) = I(z)G(z)$$

($I(z)$ sum of barely connected diagrams)

Conjecture (Weinberg, 1964). $I(z)$ compact where convergent.

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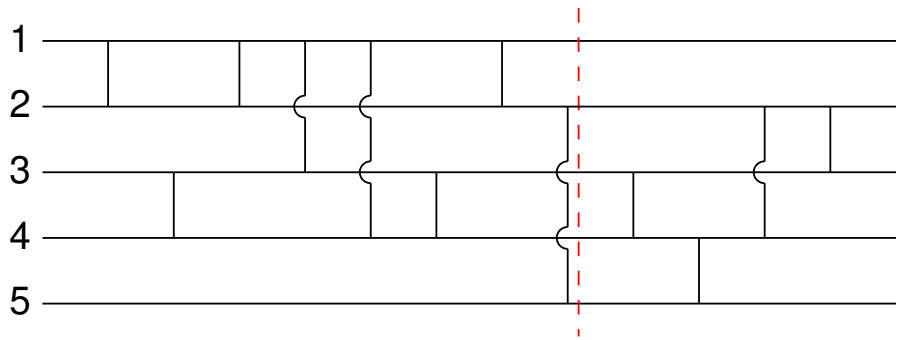

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But where?

Hunziker's resummation (1964)

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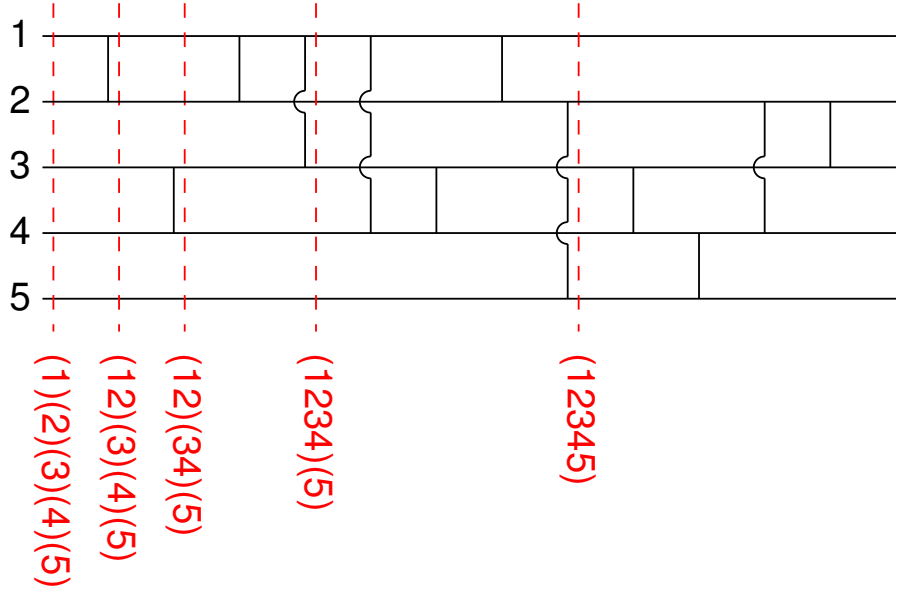

String $S = (a_N, \ldots a_1)$: a sequence of cluster decompositions a_k having k clusters proceeding by mergers of two clusters.

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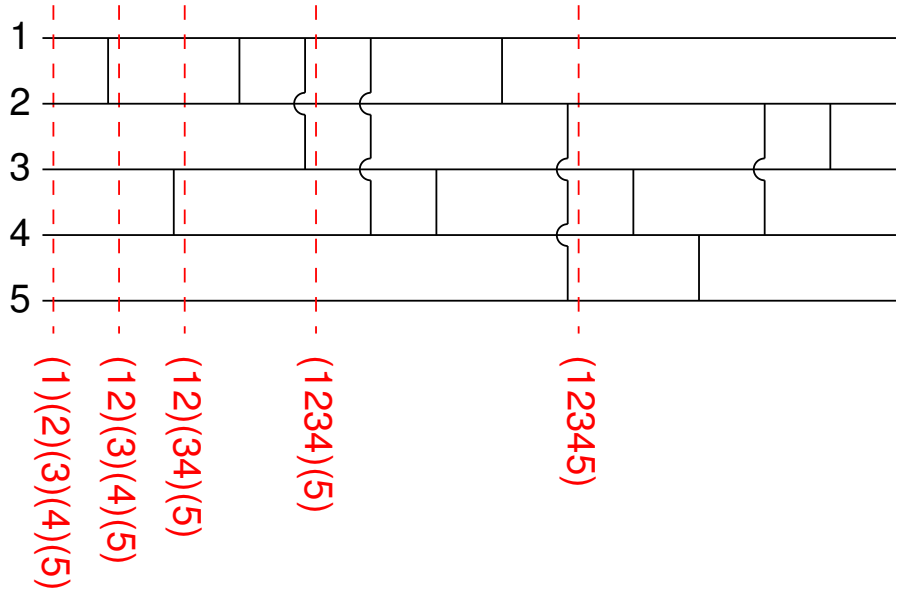

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$$C(z) = \sum_S G_{a_N}(z) V_{a_N a_{N-1}} G_{a_{N-1}}(z) \dots G_{a_2}(z) V_{a_2 a_1} G_{\cancel{a_1}}(z)$$

with $G_a(z) = (z - H_a)^{-1}$ and $V_{a_k a_{k-1}}$ potentials linking clusters of a_k but not of a_{k-1}.

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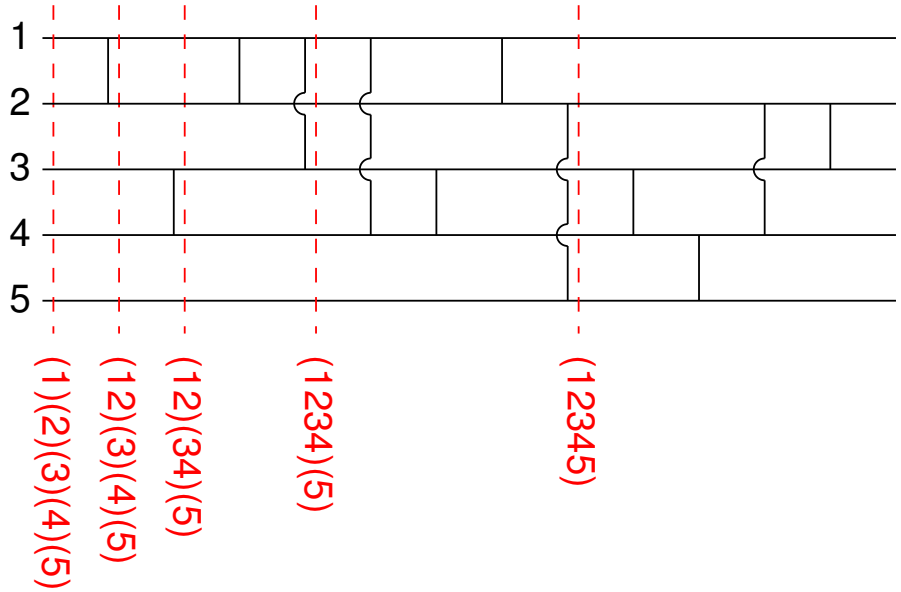

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$$I(z)\cancel{G(z)} = \sum_S G_{a_N}(z) V_{a_N a_{N-1}} G_{a_{N-1}}(z) \dots G_{a_2}(z) V_{a_2 a_1} \cancel{G(z)}$$

with $G_a(z) = (z - H_a)^{-1}$ and $V_{a_k a_{k-1}}$ potentials linking clusters of a_k but not of a_{k-1}. Alike formulae for $I(z)$, $D(z)$.

Hunziker's resummation (continued)

$$(1 - I(z))G(z) = D(z)$$

$$I(z) = \sum_{S} G_{a_N}(z) V_{a_N a_{N-1}} G_{a_{N-1}}(z) \dots G_{a_2}(z) V_{a_2 a_1}$$

$$D(z) = \sum_{S;k>1} G_{a_N}(z) V_{a_N a_{N-1}} G_{a_{N-1}}(z) \dots G_{a_{k+1}}(z) V_{a_{k+1} a_k} G_{a_k}(z)$$

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- $I(z)$ compact there

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- finite sums defined for $z \notin \bigcup_{a \neq (1 \dots N)} \sigma(H_a)$
- $I(z)$ compact there
- hence $G(z) = (z - H)^{-1}$ meromorphic there $\square$