On the Construction of Two-dimensional Models in Local Quantum Physics

(joint work with J. Barata & J. Mund)

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...but what are the basic observables? Obviously the essential instruments in high energy physics are detectors. The task of a detector is to give a signal from a specified region in space at some time. ...My conclusion was that the theory must give us for each region of space-time an algebra corresponding to the set of all observables or operations pertaining to the region. This correspondence between space-time regions and algebras is the content of the theory; nothing more nor less. ...In the case of a field theory the algebra of a region is generated by the fields smeared out by test functions with support in the region. **But there may be other possibilities of construction.**"

First surprise - Quantization is not needed

The work I will present today is formulated in a purely operator algebraic language. The observales generate a net of local von Neumann algebras, and the physically relevant information can be obtained from the net of local algebras, without ever talking of Lagrangians, classical fields, differential equations, specific observables, quantization methods or quantum fields.

Second surprise - New two-dimensional models

Many well-known models can be identified within our framework, but in addition we encounter an enormous variety of *new* two dimensional quantum theories. We currently do not know if the new quantum theories have classical limits, nor if they can be constructed from (yet unknown) Lagrangian (using conventional methods).

Geometry: Two-dimensional de Sitter space

- De Sitter space

$$dS_r \doteq \left\{ x \in \mathbb{R}^{1+2} \mid x \cdot x = x_0^2 - x_1^2 - x_2^2 = -r \right\}, \quad dS = dS_1,$$

- Wedges: set $W_1 \doteq \left\{ x \in dS \mid x_2 > |x_0| \right\}$,

$$W = \Lambda W_1 \subset dS, \qquad \Lambda \in SO_0(1,2).$$

  The set of all wedges is denoted by $\mathcal{W}$.

- Boosts

$$\Lambda_W(t) = \Lambda \Lambda_1(t) \Lambda^{-1}, \quad \Lambda_1(t) \doteq \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & 1 & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix}.$$

Wedge

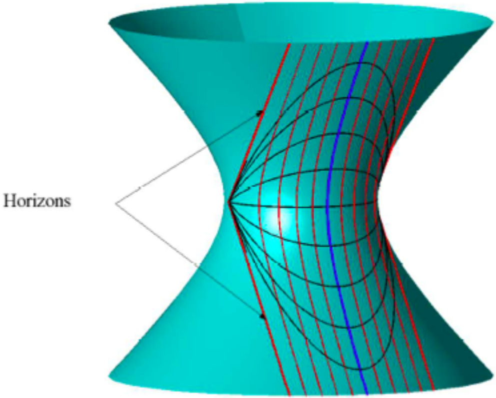


Figure : Wedge

Figures in this talk are reproduced from a talk by Hugo Moschella.

- $\Lambda_W(t)W = W$, $t \in \mathbb{R}$, and, for all $t \in \mathbb{R}$,

$$\Lambda_{\Lambda' W}(t) = \begin{cases} \Lambda' \Lambda_W(t) \Lambda'^{-1} & \text{if } \Lambda' \in SO_0(1,2) \\ \Lambda' \Lambda_W(-t) \Lambda'^{-1} & \text{if } \Lambda' \in O_+^{\downarrow}(1,2)\,. \end{cases} ,$$

- Rotations

$$\alpha \mapsto R_0(\alpha) \doteq \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} , \quad \alpha \in [0, 2\pi)\,.$$

- Horospheric Translations

$$q \mapsto D(q) \doteq \begin{pmatrix} 1 + \frac{q^2}{2} & q & \frac{q^2}{2} \\ q & 1 & q \\ -\frac{q^2}{2} & -q & 1 - \frac{q^2}{2} \end{pmatrix} , \quad q \in \mathbb{R}\,.$$

Rotations and Horospheric Translations

Cauchy Surfaces Horospheres

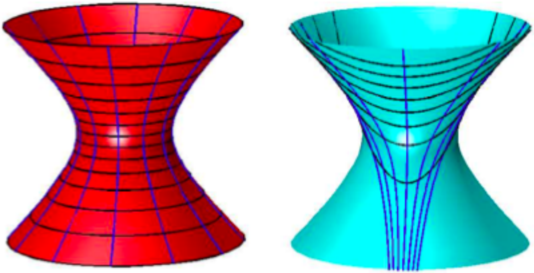


Figure : $dS \doteq \left\{ x \in \mathbb{R}^{1+2} \mid x_0^2 - x_1^2 - x_2^2 = -r^2 \right\}, \quad r > 0.$

- Time and Space Reflections

$$T \doteq \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_1 \doteq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in O(1,2).$$

- Reflection at the Edge of the Wedge

$$\Theta_{\Lambda W_1} = \Lambda (P_1 T) \Lambda^{-1}, \quad W = \Lambda W_1, \quad \Lambda \in SO_0(1,2).$$

We have

$$\Theta_W W = W', \qquad \Theta_W \mathcal{W} = \mathcal{W}.$$

Part I : Free Quantum Theories on dS

(work by Brunetti, Guido and Longo)

Modular Localization

Let $\Lambda \mapsto u(\Lambda)$ be a unitary irreducible representations of the Lorentz group $O(1,2)$ on some Hilbert space $\mathcal{H}$. Let ℓ_W be the self-adjoint generator of the one-parameter subgroup

$$t \mapsto u\big(\Lambda_W\big(\tfrac{t}{r}\big)\big).$$

Set

$$\delta_W \doteq \mathrm{e}^{-2\pi r \ell_W}, \qquad j_W \doteq u(\Theta_W).$$

δ_W is a densely defined, closed, positive non-singular linear operator on $\mathcal{H}$; j_W is an anti-unitary operator on $\mathcal{H}$.

These properties allow one to introduce the operator

$$s_W \doteq j_W \delta_W^{1/2} \,,$$

s_W is a densely defined, antilinear, closed operator on $\mathcal{H}$ with $\mathscr{R}(s_W) = \mathscr{D}(s_W)$ and $s_W^2 \subset \mathbb{1}$. Moreover,

$$u(\Lambda) s_W u(\Lambda)^{-1} = s_{\Lambda W} \,, \quad \Lambda \in SO_0(1,2).$$

Definition (Brunetti, Guido, Longo, 2002)

The *modular localisation map* $W \mapsto \mathcal{H}(W)$ associates an $\mathbb{R}$-linear subspace

$$\mathcal{H}(W) \doteq \{ h \in \mathscr{D}(s_W) \mid s_W h = h \}$$

of $\mathcal{H}$ to a wedge $W \in \mathcal{W}$.

Theorem (Brunetti, Guido, Longo, 2002)

Each $\mathcal{H}(W)$ is an $\mathbb{R}$-linear, closed, standard subspace in $\mathcal{H}$. Moreover, s_W is the Tomita operator of $\mathcal{H}(W)$, i.e.,

$$\begin{aligned} s_W : \mathcal{H}(W) + i\mathcal{H}(W) &\to \mathcal{H}(W) + i\mathcal{H}(W) \\ h + ik &\mapsto h - ik. \end{aligned}$$

In particular,

$$\delta_W^{it}\mathcal{H}(W) = \mathcal{H}(W) \quad \text{and} \quad j_W\mathcal{H}(W) = \mathcal{H}(W)',$$

with $\mathcal{H}(W)'$ the symplectic complement of $\mathcal{H}(W)$ in $\mathcal{H}$. Moreover,

$$u(\Lambda)\mathcal{H}(W) = \mathcal{H}(\Lambda W), \qquad \Lambda \in SO_0(1,2).$$

Fock space

- Fock space $e^{\mathcal{H}} \doteq \oplus_{n=0}^{\infty} \mathcal{H}^{\otimes_s^n}$,
- Coherent vectors

$$e^h = \oplus_{n=0}^{\infty} \underbrace{h \otimes_s \cdots \otimes_s h}_{n-times}$$

- Exponentiation of operators: A a closed densely defined linear operator on $\mathcal{H}$. Then

$$e^A \colon \mathscr{H} \to \mathscr{H}$$

  is the closure of the linear operator acting on the linear combinations of coherent vectors with exponent in $\mathscr{D}(A)$ such that:

$$e^A e^h = e^{Ah}.$$

  Exponentiation preserves self-adjointness, positivity and unitarity.

Weyl algebra

For $h, g \in \mathcal{H}$, the relations

$$V(h)V(g) = \mathrm{e}^{-i\Im\langle h,g\rangle} V(h+g),$$
$$V(h)\Omega_\circ = \mathrm{e}^{-\frac{1}{2}||f||^2} \mathrm{e}^{ih},$$

define unitary operators, called the Weyl operators .

They satisfy $V^*(h) = V(-h)$ and $V(0) = \mathbb{1}$. The one-parameter group $\Lambda \mapsto u(\Lambda)$ induces a group of automorphisms

$$\alpha^\circ_\Lambda(V(h)) \doteq V\big(u(\Lambda)h\big), \qquad h \in \mathcal{H}, \quad \Lambda \in SO_0(1,2),$$

representing the free dynamics.

Definition (The Net of Local Algebras)

We associate v. Neumann algebras to space-time regions in dS:

i.) for the wedge W_1,

$$\mathscr{A}_\circ(W_1) \doteq \{V(h) \mid h \in \mathcal{H}(W_1)\}'';$$

ii.) for an arbitrary wedge W, set

$$\mathscr{A}_\circ(W) \doteq \alpha^\circ_\Lambda\left(\mathscr{A}_\circ(W_1)\right), \qquad W = \Lambda W_1;$$

iii.) for an arbitrary bounded, causally complete, convex region $\mathcal{O} \subset dS$, set

$$\mathscr{A}_\circ(\mathcal{O}) = \bigcap_{\mathcal{O}\subset W} \mathscr{A}_\circ(W)\,.$$

The map $\mathcal{O} \mapsto \mathscr{A}_\circ(\mathcal{O})$ preserves inclusions, the algebras $\mathscr{A}_\circ(\mathcal{O})$ are hyperfinite type III_1 factors, and $\alpha^\circ_\Lambda\left(\mathscr{A}_\circ(\mathcal{O})\right) = \mathscr{A}_\circ(\Lambda\mathcal{O})$.

Abelian von Neumann Algebras for S^1

The time-reflection T on dS induces a conjugation ϑ on $\mathcal{H}$. The $\mathbb{R}$-linear subspace

$$\mathcal{H}_\vartheta = \{h \in \mathcal{H}(dS) \mid \vartheta h = h\}$$

is standard. It consists of time-reflection invariant functions.

Lemma

The weak closure $\mathscr{U}$ of the C^*-algebra $\mathcal{U}$ generated by the Weyl operators $\{V(h) \mid h \in \mathcal{H}_\vartheta\}$ is a maximal abelian von Neumann algebra on $\mathscr{H}$ with cyclic and separating vector $\mathrm{e}^0 = \Omega_\circ$.

The spectrum K of $\mathcal{U}$ is a (weak*) compact Hausdorff space and $C(K) \cong \mathcal{U}$. The vector $\Omega_\circ \cong 1_K$,

$$L^\infty(K, \mathrm{d}\nu) \cong \mathscr{U} \qquad \text{and} \qquad L^2(K, \mathrm{d}\nu) \cong \overline{\mathscr{U}\,\Omega_\circ} = \mathscr{H} \,.$$

A normal state ω defines an element $\omega_{\restriction \mathscr{U}}$ in $\mathscr{U}_*^+$, represented by the square of a unique positive function in $L^2(K, \mathrm{d}\nu)$.

Lemma

Let ω be a normal, rotation invariant state. Then there exists a rotation invariant, positive operator $A \in L^2(\mathscr{U}, \Omega_0)$, such that

$$\omega_{\restriction \mathscr{U}}(\,.\,) = \langle \Omega, .\,\Omega \rangle, \quad \Omega \doteq A\Omega_\circ \in \mathscr{H}, \quad \widetilde{A} \in L^2(K, \mathrm{d}\nu).$$

If $\omega_{\restriction \mathscr{U}}$ is faithful, $\widetilde{A} > 0$ a.e. and Ω is cyclic & separating for $\mathscr{U}$.

Part II: Interacting Quantum Theories on dS

Lemma

Let $\omega = \omega \circ \alpha^{\circ}_{R_0(\gamma)}$, $\gamma \in [0, 2\pi)$, *with* $\omega_{\restriction \mathscr{U}}(\,.\,) = \langle \Omega, .\, \Omega \rangle$ *faithful. Then there exists a self-adjoint operator* $V^{(0)}$ *affiliated to* $\mathscr{U}(W_1)$ *such that* $\Omega_\circ \in \mathscr{D}(u_{-\frac{i}{2}})$ *and*

$$\Omega = u_{-\frac{i}{2}} \Omega_\circ = u_{-\frac{i}{4}} J_{W_1} u_{-\frac{i}{4}} \Omega_\circ ,$$

with $u_{-\frac{i}{4}}$ *affiliated to* $\mathscr{A}_\circ(W_1)$ *and*

$$u_{i\theta} = \mathbb{1} + \sum_{n \geq 1} (-1)^n \int_0^{\theta} \mathrm{d}\theta_1 \cdots \int_0^{\theta_{n-1}} \mathrm{d}\theta_n \; \sigma^{\circ}_{i\theta_n}(V^{(0)}) \cdots \sigma^{\circ}_{i\theta_1}(V^{(0)}) .$$

$t \mapsto \sigma^{\circ}_t(\,.\,)$ *denotes the modular group for the pair* $(\mathscr{A}_\circ(W_1), \Omega_\circ)$.

Theorem

The vector $\Omega \in \mathscr{P}^{1/2}_{\Omega_\circ}(\mathscr{A}_\circ(W_1))$ *in the natural positive cone for the pair* $(\mathscr{A}_\circ(W_1), \Omega_\circ)$ *is cyclic and separating for* $\mathscr{A}_\circ(W_1)$.

The modular Δ_{W_1} operator for the pair $(\mathscr{A}_\circ(W_1), \Omega)$ gives rise to a one-parameter group

$$t \mapsto \Delta^{it}_{W_1}\,, \qquad t \in \mathbb{R}\,,$$

which leaves $\mathscr{A}_\circ(W_1)$ and Ω invariant. Since Ω is an element of the positive cone $\mathcal{P}^\sharp(\mathscr{A}_\circ(W_1), \Omega_\circ)$ we have $J_{W_1} = J^\circ_{W_1}$.

Connes cocycle (non-comm. L^p spaces, Araki & Masuda)

Since $\mathscr{H} = L^2(\mathscr{A}_\circ(W_1), \Omega_\circ)$, $\Omega = \Delta^{1/2}_{\Omega,\Omega_\circ}\Omega_\circ \in \mathscr{P}^{1/2}_{\Omega_\circ}(\mathscr{A}_\circ(W_1))$. The relative modular operator $\Delta_{\Omega,\Omega_\circ} = S^*_{\Omega,\Omega_\circ}\overline{S_{\Omega,\Omega_\circ}}$ arises from the polar decomposition of the anti-linear map

$$S_{\Omega,\Omega_\circ} M\Omega_\circ = M^*\Omega\,, \qquad M \in \mathscr{A}_\circ(W_1)\,.$$

$\exists$ strongly continuous one-parameter family of unitaries

$$u_t = [D\omega : D\omega_\circ]_t = \Delta^{it}_{\Omega,\Omega_\circ}\Delta_\circ^{-it} \in \mathscr{A}_\circ(W_1)\,, \quad t \in \mathbb{R}\,,$$

which intertwines the modular groups for ω and $\omega_\circ$, *i.e.*,

$$\sigma_t(M) = u_t\sigma^\circ_t(M)u_t^*\,, \quad \forall M \in \mathscr{A}_\circ(W_1),$$

and satisfies the cocycle relation $u_{t+s} = u_t\sigma^\circ_t(u_s)$, $t, s \in \mathbb{R}$.

Interacting Automorphisms

Theorem (inspired by Osterwalder, Fröhlich and Seiler)

The boost $t \mapsto \Delta^{it}_{W_1}$ *and the (free) rotations* $U^{\circ}(R_0(\alpha))$, $\alpha \in [0, 2\pi)$, *generate a representation* $U(\Lambda)$ *of* $SO_0(1,2)$.

Definition

The unitary representation $\Lambda \mapsto U(\Lambda)$ induces a group of automorphisms

$$\alpha_\Lambda(V(h)) \doteq U(\Lambda)V(h)U(\Lambda)^{-1}, \qquad h \in \mathcal{H}, \quad \Lambda \in O(1,2),$$

representing the *interacting dynamics*.

Definition (The Net of Local Algebras)

We proceed just as for the free theory:

i.) for the wedge W_1, set $\mathscr{A}(W_1) \doteq \mathscr{A}_\circ(W_1)$;

ii.) for an arbitrary wedge W, set

$$\mathscr{A}(W) \doteq \alpha_\Lambda \left(\mathscr{A}(W_1) \right) , \qquad W = \Lambda W_1 ;$$

iii.) for a causally complete, convex region $\mathcal{O} \subset dS$, set

$$\mathscr{A}(\mathcal{O}) = \bigcap_{\mathcal{O} \subset W} \mathscr{A}(W) .$$

The map $\mathcal{O} \mapsto \mathscr{A}(\mathcal{O})$ is the net of local von Neumann algebras for the interacting theory.

Theorem (Verification of the Haag-Kastler Axioms)

The representation $\alpha \colon \Lambda \mapsto \alpha_\Lambda$ of the Lorentz group $SO_0(1,2)$ is covariant:

$$\alpha_\Lambda\big(\mathscr{A}(\mathcal{O})\big) = \mathscr{A}(\Lambda\mathcal{O})\,, \qquad \Lambda \in SO_0(1,2)\,.$$

The local algebras satisfy micro-causality, i.e.,

$$\mathscr{A}(\mathcal{O}_1) \subset \mathscr{A}(\mathcal{O}_2)' \quad \textit{if} \quad \mathcal{O}_1 \subset \mathcal{O}_2'\,.$$

Here $\mathcal{O}'$ denotes the space-like complement of $\mathcal{O}$ in dS and $\mathscr{A}(\mathcal{O})'$ is the commutant of $\mathscr{A}(\mathcal{O})$ in $\mathcal{B}(\mathcal{H})$.

Theorem (continued; Barata, Mund & J. (2013))

The unit vector $\Omega \in \mathcal{H}$, describing the de Sitter vacuum, is the unique (up to a phase) vector, which

- *is invariant under the action of $U(SO_0(1,2))$;*
- *satisfies the geodesic KMS condition of Borchers and Buchholz: for every wedge W*

$$\omega_{\restriction \mathscr{A}(W)}(A) \doteq \langle \Omega, A\Omega \rangle \,, \qquad A \in \mathscr{A}(W)\,,$$

satisfies the KMS-condition at inverse temperature $2\pi r$ with respect to the group $t \mapsto \alpha_{\Lambda_W(t)}$, $t \in \mathbb{R}$.

Theorem (Borchers & Buchholz)

For any open region $\mathcal{O} \subset dS$ there holds

$$\overline{\mathscr{A}(\mathcal{O})\Omega} = \mathcal{H}.$$

Theorem (Barata, Mund & J. (2013))

Let

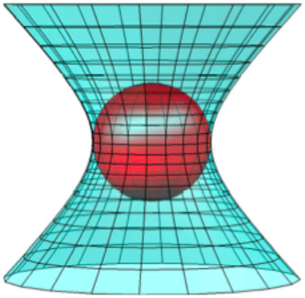
$$I(\alpha, t) \doteq S^1 \cap \Big(\bigcup_{y \in \Lambda^{(\alpha)}(t) I} \Gamma^-(y) \cup \Gamma^+(y) \Big).$$

It follows that for $t \neq 0$, one has the embedding

$$\alpha_{\Lambda^{(\alpha)}(t)}(\mathscr{A}(\mathcal{O}_I)) \hookrightarrow \mathscr{A}_\circ\big(\Lambda^{(\alpha)}(t)\mathcal{O}_{I(\alpha,t)}\big).$$

This result is related to Borcher's notion of relative locality .

The $P(\varphi)_2$ model (Glimm & Jaffe; Figari, Høegh-Krohn & Nappi; Klein & Landau; Gérard & J. ; Barata, Mund & J.)



can be reconstructed from Markov processes on the Euclidean sphere.

$$\Omega = \frac{\mathcal{V}\big(\mathrm{e}^{-V(\overline{S_+})}\big)}{\|\mathcal{V}\big(\mathrm{e}^{-V(\overline{S_+})}\big)\|} = \frac{T\mathrm{e}^{-\int_0^{1/2}\mathrm{d}\theta\,\sigma^\circ_{i\theta}(V_0(\cos_\psi))}\Omega_\circ}{\|T\mathrm{e}^{-\int_0^{1/2}\mathrm{d}\theta\,\sigma^\circ_{i\theta}(V_0(\cos_\psi))}\Omega_\circ\|},$$

with

$$V_0(h) = \int_0^\pi \mathrm{d}\psi\, h(\psi) :\mathscr{P}(\Phi(0,\psi)):_{C_0}.$$

Part III: From de Sitter to Minkowski Space

Geometry

Consider the mapping

$$\mathbb{R}^{1+1} \ni (t,x) \mapsto \xi_r(t,x) \doteq D\left(\tfrac{x}{r}\right) \Lambda_1\left(\tfrac{t}{r}\right) \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix}.$$

The points $\xi_r(t,x) + \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix}$ are in the interior of $\Gamma^+(W_1) \subset dS_r$.

Clearly,

$$\xi_r(t,x) \to \begin{pmatrix} t \\ x \\ 0 \end{pmatrix}$$

uniformly (in the Euclidean norm on $\mathbb{R}^3$) on compact sets containing the origin, as $r \to \infty$.

Next approximate the Poincaré group by the Lorentz group in one dimension more.

Lemma (Takahashi-Hannabus)

Almost every element $g \in SO_0(1,2)$ can be written uniquely in the form of a product

$$g = \Lambda_2(s) P^k \Lambda_1(t) D(q)$$

with $s, t, q \in \mathbb{R}$, $k = \{0, 1\}$ and $P = R_0(\pi)$ the spatial reflection.

The spatial reflection is necessary to account for rotations $R_0(\alpha)$, $\frac{\pi}{2} < \alpha < \frac{3\pi}{2}$, in the Iwasawa decomposition.

Lemma

Let $\mathcal{N} \subset SO_0(1,2)$ *be a neighbourhood of the unit. The map* $\Pi_r : \mathcal{N} \to E_0(1,1)$ *given by*

$$\Lambda_2\left(\tfrac{s}{r}\right)P^0\Lambda_1\left(\tfrac{t}{r}\right)D\left(\tfrac{x}{r}\right) \mapsto \Lambda_2(s)T(t,x),$$

with $T(t,x)$ *a translation on* $\mathbb{R}^{1+1}$*, defines a contraction of the group* $SO_0(1,2)$ *to* $E_0(1,1)$*. In particular,*

$$g \circ g' = \lim_{r\to\infty} \Pi_r\left(\Pi_r^{-1}(g) \circ \Pi_r^{-1}(g')\right), \qquad \forall g, g' \in E_0(1,1).$$

By construction, $g \cdot x = \lim_{r\to\infty} \xi_r^{-1}\left(\Pi_r^{-1}(g) \cdot \xi_r(x)\right)$ for $g \in E_0(1,1)$ and $x \in \mathbb{R}^{1+1}$.

The unitary irreducible representations u_ν of $SO_0(1,2)$ within the principle series all act on the Hilbert space

$$\mathcal{H} \cong L^2\big(\mathbb{R}, \tfrac{\mathrm{d}k}{2\sqrt{k^2+m^2}}\big) \oplus L^2\big(\mathbb{R}, \tfrac{\mathrm{d}k}{2\sqrt{k^2+m^2}}\big) \equiv \mathcal{H}_+ \oplus \mathcal{H}_- \, .$$

Moreover, each component in this direct sum carries a unitary irreducible representation $\mathscr{D}_m$ of the Poincaré group for mass m given by

$$\big(\mathscr{D}_m(\Lambda_2(s)T(t,q))h\big)(k) = \mathrm{e}^{i(t,q)\cdot(\sqrt{k^2+m^2},k)} h(k+k'),$$

with $(t,q) \in \mathbb{R}^{1+1}$ and $m \sinh s = k'$.

Theorem (related to work of Mickelsson and Niederle)

Consider a unitary irreducible representation u_ν *of* $SO_0(1,2)$, $\nu = mr$. *Let* $g \in \mathcal{H}$. *Then*

$$\lim_{r\to\infty} \left|\left| \Big(u_{mr}\left(\Lambda_2\left(\tfrac{s}{r}\right)\Lambda_1\left(\tfrac{t}{r}\right)D\left(\tfrac{q}{r}\right)\right) - \mathcal{D}_m\left(\Lambda_2(s)T(t,q)\right)\Big) g \right|\right|_{\mathcal{H}} = 0 \,.$$

$\mathcal{D}_m$ *is a reducible representation of the Poincaré group* $E_0(1,1)$, *given by*

$$\mathcal{D}_m = \mathscr{D}_m \oplus \mathscr{D}_{-m} \,.$$

Convergence of Weyl operators

Let $\widetilde{h}_\infty \in C_0^\infty(\mathbb{R}^{1+1})$ with support of $\widetilde{h}_\infty$ in some double cone $\mathcal{O}$. Define

$$\widetilde{h}_r(\xi_r(t,x)) \doteq \widetilde{h}_\infty(t,x)$$

with compact support in dS_r. It follows that

$$s\text{-}\lim_{r\to\infty} V(h_r) = V(h_\infty)\,,$$

as

$$\lim_{r\to\infty} || \int_{dS} \mathrm{d}\mu_{dS}(\xi_r)\, \widetilde{h}(\xi_r) \left(\tfrac{\xi_r}{r}\cdot\tfrac{p}{m}\right)^{-\frac{1}{2}+imr}$$

$$- \int \mathrm{d}t\mathrm{d}x\, \widetilde{h}_\infty(t,x)\, \mathrm{e}^{i(t,x)\cdot(\sqrt{k^2+m^2},k)} || = 0\,, \quad p = \begin{pmatrix} \sqrt{k^2+m^2} \\ k \\ m \end{pmatrix}.$$

Lemma (Convergence of Vacuum States)

The weak-limit of the net* $\{\omega_r^\circ\}_{r>0}$ *as* $r \to \infty$ *coincides with the restriction of the Fock vacuum* $\omega_\circ$ *to* $\mathscr{B}(\mathscr{H}_+) \otimes \mathbb{1}$. *It is invariant under the action of the Poincaré group, i.e.,*

$$\omega_\infty^\circ \circ \alpha_g^{\circ(\infty)} = \omega_\infty^\circ, \qquad g \in E_0(1,1),$$

and satisfies the spectrum condition, i.e., for two strictly local elements A, B *the function*

$$(t,q) \mapsto \omega_\infty\big(A\alpha_{T(t,q)}^{\circ(\infty)}(B)\big)$$

allows an analytic continuation into the tube

$$\mathcal{T}_+ = \{(t,q) \in \mathbb{C}^2 \mid \Im|q| < \Im t\}.$$

In summary, we have reconstructed the scalar free field of mass m on Minkowski space starting from free quantum theories on de Sitter spaces of increasing radius r.

Theorem (Verification of the Haag-Kastler Axioms)

Let

a.) $\mathscr{A}_\circ^{(\infty)}(\mathcal{O})$ *be the weak limit of* $\mathscr{A}_\circ^{(r)}\big((\xi_r^{-1}\mathcal{O})''\big)$;

b.) $\alpha_g^{\circ(\infty)}$ *be a weak limit of* $\alpha^{\circ(r)}_{\Pi_r^{-1}(g)}$, $g \in E_0(1,1)$;

c.) ω_∞° *be a weak* limit of* $\{\omega_r^\circ\}_{\frac{1}{r}>0}$

as the radius $r \to \infty$.

Theorem (continued)

It follows that

i.) (Isotony). *The map* $\mathcal{O} \to \mathscr{A}_\circ^{(\infty)}(\mathcal{O})$ *from the set of open, bounded, contractible regions* $\mathcal{O} \subset \mathbb{R}^{1+1}$ *to unital von Neumann algebras*

$$\mathscr{A}_\circ^{(\infty)}(\mathcal{O}) \in \mathscr{B}(\mathscr{H}_+) \otimes \mathbb{1}$$

preserves inclusions;

ii.) (Microcausality). $\mathscr{A}_\circ^{(\infty)}(\mathcal{O}_1) \subset \mathscr{A}_\circ^{(\infty)}(\mathcal{O}_2)'$ *if* $\mathcal{O}_1 \subset \mathcal{O}_2'$.

$\mathcal{O}'$ *is the space-like complement of* $\mathcal{O}$ *in dS;*
$\mathscr{A}_\circ^{(\infty)}(\mathcal{O})'$ *is the commutant of* $\mathscr{A}_\circ^{(\infty)}(\mathcal{O})$ *in* $\mathscr{B}(\mathscr{H}_+) \otimes \mathbb{1}$;

Theorem (continued)

iii.) (Covariance). *The automorphisms* $\alpha^{\circ(\infty)} : g \mapsto \alpha_g^{\circ(\infty)}$ *provides a representation of* $E_0(1,1)$

$$g \to \alpha_g^{\circ(\infty)} .$$

Moreover, they act geometrically, i.e.,

$$\alpha_g^{\circ(\infty)}\big(\mathscr{A}_\circ^{(\infty)}(\mathcal{O})\big) = \mathscr{A}_\circ^{(\infty)}(g\mathcal{O}) , \quad g \in E_0(1,1) .$$

iv.) (Existence of vacuum states). *The weak* accumulation point* ω_∞°

— *is invariant under the action of* $E(1,1)$*, i.e.,*

$$\omega_\infty^\circ \circ \alpha_g^{\circ(\infty)} = \omega_\infty^\circ , \qquad g \in E(1,1) ,$$

Theorem (continued)

iv.) — *satisfies the spectral condition: for two strictly local elements* $A, B \in \mathscr{A}_\circ^{(\infty)}(\mathcal{O})$, $\mathcal{O} \subset \mathbb{R}^{1+1}$, *the function*

$$(t, q) \mapsto \omega_\infty^\circ \left(A \alpha_{T(t,q)}^{\circ(\infty)}(B) \right)$$

allows an analytic continuation to the tube

$$\mathcal{T}_+ = \{(t, q) \in \mathbb{C}^2 \mid \Im |q| < \Im t\} .$$

But what about the interacting cases?

Lemma (Convergence of time-zero Double Cones)

Let $\mathcal{O}_I$ be a double cone with base I on the time-zero line. Then

$$\mathscr{A}_{(\infty)}(\mathcal{O}_I) = \mathscr{A}_{\circ}^{(\infty)}(\mathcal{O}_I).$$

Theorem (Convergence of Vacuum States)

The weak* accumulation point ω_∞ satisfies the spectral condition: for every two elements $A, B \in \mathscr{A}_\infty(\mathbb{R}^{1+1})$, the function

$$(t, q) \mapsto \omega_\infty\big(A\alpha^{(\infty)}_{T(t,q)}(B)\big)$$

allows an analytic continuation to the tube

$$\mathcal{T}_+ = \{(t, q) \in \mathbb{C}^2 \mid \Im|q| < \Im t\}\,.$$

But what about the automorphisms?

In case of the $\mathscr{P}(\varphi)_2$ model explicit computations (using finite speed of light) show the existence of the limit for a fixed automorphism and a strictly local element as $r \to \infty$, following the work of Glimm and Jaffe.

In the abstract case, the scaling algebras of Buchholz and Verch should provide a general framework, which hopefully will allow us to demonstrate that the modular localisation on the de Sitter space goes over to the modular localisation on Minkowski space. (This final part is work in progress.)