# Random Band Matrices and the Extended States Conjecture 

Antti Knowles<br>Courant - NYU

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With L. Erdős, H.T. Yau, and J. Yin

## Part I

Band matrices and the Anderson transition

A quantum system of sufficient complexity exhibits one of the two following behaviours.
(A) Delocalized eigenvectors, local spectral statistics given by RMT. (Weak disorder)
(B) Localized eigenvectors, local spectral statistics are Poisson. (Strong disorder)

System is described by a Hamiltonian $H$ which has to be 'generic' to yield sufficient complexity.
Popular mathematical playgrounds for probing the universality conjecture:

- Random matrices (Wigner matrices, random Schrödinger operators)
- Quantum chaos models

For simplicity, work on one-dimensional lattice $\{1, \ldots, L\}$ with $L \rightarrow \infty$.

Wigner random matrix. The entries of $H$ are i.i.d. up to the constraint $H=H^{*}$. This is a mean-field model with no spatial structure.
Behaviour of type (A).
(Erdős-Schlein-Yau-. . [2009-2012], Tao-Vu [2009-2012])
Random Schrödinger operator. On-site randomness + short-range hopping.

$$
H=-\Delta+\sum_{x} v_{x}=\left(\begin{array}{ccccc}
v_{1} & 1 & & & \\
1 & v_{2} & 1 & & \\
& 1 & \ddots & \ddots & \\
& & \ddots & v_{L-1} & 1 \\
& & & 1 & v_{L}
\end{array}\right)
$$

Behaviour of type (B).
(Goldscheid-Molchanov-Pastur [1997], Minami [1996])

- A family of interpolating models.
- Allow a precise statement of Wigner's conjecture, together with a transition from (A) to (B).

Let $H=\left(H_{x y}\right)$ be an $L \times L$ matrix with mean-zero entries independent up to contraint $H=H^{*}$. Let $f$ be a probability density on $\mathbb{R}$ (the band profile) and suppose that

$$
\mathbb{E}\left|H_{x y}\right|^{2}=S_{x y}:=\frac{1}{W} f\left(\frac{x-y}{W}\right) .
$$

Here $W \in[1, L]$ is the band width.

$$
(\infty)
$$

Varying $W$ provides a means to test the transition from (A) to (B).

Conjecture (Fyodorov-Mirlin [1991])
The transition occurs at $W \sim L^{1 / 2}$.

Let $\ell$ denote the typical localization length of the eigenvectors of $H$. Then the conjecture means that $\ell \sim W^{2}$.

- Upper bound: $\ell \leqslant W^{8}$ (Schenker [2010]).
- Lower bound: arises from quantum diffusion.

Define the expected quantum transition probability from 0 to $x$ in time $t$ through

$$
\varrho(t, x):=\mathbb{E}\left|\left(\mathrm{e}^{-\mathrm{i} t H}\right)_{x 0}\right|^{2} .
$$

Consider the diffusive regime $t=\zeta T$ and $x=\zeta^{1 / 2} W X$ with $\zeta \rightarrow \infty$.
Diffusion cannot hold for $x \gg W^{2} \Longrightarrow$ choose $\zeta=W^{\kappa}$ for $0<\kappa<2$.

## Theorem (Erdős-K [2011])

Fix $0<\kappa<1 / 3$. Then for all $T \geqslant 0$

$$
\varrho\left(W^{\kappa} T, W^{1+\kappa / 2} X\right) \longrightarrow \int_{0}^{1} \frac{4}{\pi} \frac{\lambda^{2}}{\sqrt{1-\lambda^{2}}} G(\lambda T, X) \mathrm{d} \lambda
$$

weakly in $X$, where

$$
G(T, X):=\frac{1}{\sqrt{2 \pi T D}} \mathrm{e}^{-\frac{1}{2 T D} X^{2}}, \quad D:=\frac{1}{2} \int X^{2} f(X) \mathrm{d} X .
$$

Corollary: eigenvectors have localization length $\ell \geqslant W^{1+1 / 6}$.

Instead of studying the unitary time evolution $\mathrm{e}^{-\mathrm{i} t H}$, consider the resolvent $G(z):=(H-z)^{-1}$. These are equivalent by the identities

$$
G(z)=\mathrm{i} \int_{0}^{\infty} \mathrm{e}^{\mathrm{i} t z} \mathrm{e}^{-\mathrm{i} t H} \mathrm{~d} t, \quad \mathrm{e}^{-\mathrm{i} t H}=\frac{\mathrm{i}}{2 \pi} \oint \mathrm{e}^{-\mathrm{i} t z} G(z) \mathrm{d} z
$$

Control of $\mathrm{e}^{\mathrm{i} t H} \quad \Longleftrightarrow \quad$ Control of $G(E+\mathrm{i} \eta)$ for $\eta \sim t^{-1}$.
Theorem (Erdős-K-Yau-Yin [2012])
Suppose that $L \leqslant W^{5 / 4}$ and $\eta \geqslant W^{-1 / 2}$. Then, after a small amount of averaging, the matrix $\left(\left|G_{x y}\right|^{2}\right)$ is with high probability equal to $\Theta(1+o(1))$, where

$$
\Theta:=\frac{|m|^{2} S}{1-|m|^{2} S} ;
$$

here $S:=\left(S_{x y}\right)$ is the step distribution of a random walk and $m$ is the Stieltijes transform of Wigner's semicircle law.

Corollary: For $L \leqslant W^{5 / 4}$ the eigenvectors are delocalized with high probability.

In Fourier space $\Theta$ reads

$$
\frac{|m|^{2} \widehat{S}(p)}{1-|m|^{2} \widehat{S}(p)} \approx \frac{\gamma}{\eta+D \gamma(W p)^{2}}, \quad \gamma:=\frac{\sqrt{4-E^{2}}}{2}
$$

where $\widehat{S}(p)$ is the Fourier transform of $S_{x 0}$.

This is the resolvent of the classical diffusion operator $D \gamma(W p)^{2}$.

Going back to $x$-space we get
$\eta \Theta_{x y} \approx \begin{cases}\frac{\sqrt{\eta}}{W} \exp \left(-\sqrt{\eta} \frac{|x-y|}{W}\right) & \text { if } \eta \geqslant\left(\frac{W}{L}\right)^{2} \\ \frac{1}{L} & \text { if } \eta \leqslant\left(\frac{W}{L}\right)^{2} .\end{cases}$


Part II
Statistics of eigenvalue density

## Smoothed eigenvalue density on mesoscopic scales

From now on, allow arbitrary spatial dimension $d \geqslant 1$ : indices $x, y$ lie in the discrete $d$-dimensional cube of side length $L$.
Let $\lambda_{1}, \ldots, \lambda_{L^{d}}$ denote the eigenvalues of $H / 2$.
Define the smoothed eigenvalue density at energy $E \in(-1,1)$ on scale $\eta$ through

$$
Y(E):=\sum_{\alpha} \frac{1}{\eta} \phi\left(\frac{\lambda_{\alpha}-E}{\eta}\right),
$$

where $\phi$ is smooth and has sufficient decay.


Goal: compute the (normalized) covariance

$$
Z\left(E_{1}, E_{2}\right):=\frac{\mathbb{E}\left(Y\left(E_{1}\right) Y\left(E_{2}\right)\right)-\mathbb{E} Y\left(E_{1}\right) \mathbb{E} Y\left(E_{2}\right)}{\mathbb{E} Y\left(E_{1}\right) \mathbb{E} Y\left(E_{2}\right)}
$$

in the mesoscopic regime $L^{-d} \ll \eta \ll 1$.
Two key questions:
(a) Density fluctuations, $Z(E, E)$.
(b) Decay of correlations $Z\left(E_{1}, E_{2}\right)$ in $E_{2}-E_{1} \gg \eta$.

Let $\left\{\lambda_{\alpha}\right\}$ be a stationary Poisson point process with intensity $L^{d}$. Then

$$
Z\left(E_{1}, E_{2}\right)=\frac{C}{L^{d} \eta} \int \phi(x) \phi\left(x-\frac{E_{2}-E_{1}}{\eta}\right) \mathrm{d} x .
$$

In particular:
(a) The density fluctuations behave according to

$$
Z(E, E)=\frac{C}{L^{d} \eta} .
$$

(b) The correlations $Z(E+\Delta / 2, E-\Delta / 2)$ decay in $\Delta$ according to the tail of $\phi$.

Computation by Altshuler and Shklovskii [1986]. Suppose that $\eta \geqslant(W / L)^{2}$.
(a) For $d=1,2,3$ the density fluctuations satisfy

$$
Z(E, E) \sim \frac{C_{d}}{\sqrt{\operatorname{det} D}\left(1-E^{2}\right)^{2}}\left(\frac{1}{L W}\right)^{d}\left(\frac{\eta}{\sqrt{1-E^{2}}}\right)^{d / 2-2}
$$

where $C_{d}>0$.
(b) For $d=1,3$ and $\Delta \gg \eta$ the correlations satisfy

$$
\begin{aligned}
& Z\left(E+\frac{\Delta}{2}, E-\frac{\Delta}{2}\right) \sim \frac{K_{d}}{\sqrt{\operatorname{det} D}\left(1-E^{2}\right)^{2}}\left(\frac{1}{L W}\right)^{d}\left(\frac{\Delta}{\sqrt{1-E^{2}}}\right)^{d / 2-2} \\
& \text { where } K_{1}<0 \text { and } K_{3}>0
\end{aligned}
$$

Focus on the regime $\eta \geqslant W^{2} / L^{2}$ where boundary effects are irrelevant for the diffusion (choose a sufficiently large sample).

- For which $\eta$ do the statistics of $Y(E)$ become Poisson? At the transition to/from Poisson, the variance of $Y(E)$ must have the magnitude

$$
\left.\left.\frac{1}{L^{d} W^{d}} \eta^{d / 2-2} \sim Z(E, E)\right|_{\mathrm{band}} \stackrel{!}{\sim} Z(E, E)\right|_{\text {Poisson }} \sim \frac{1}{L^{d} \eta},
$$

i.e.

$$
\frac{1}{W^{d}} \sim \eta^{1-d / 2}
$$

Conclusion: expect transition only for $d=1$, at $\eta \sim W^{-2}$.

- What about the localization length $\ell$ for $d=1$ ? $L \eta$ randomly chosen eigenvalues have overlapping eigenvectors if and only if $\eta \gg \ell^{-1}$. Hence the disappearance of Poisson statistics (emergence of eigenvalue correlations) should occur at $\eta \sim \ell^{-1}$. This yields $\ell \sim \eta^{-1} \sim W^{2}$ at the transition.


## Proof of the AS formulae

## Theorem (Erdős-K [2013])

- The AS formulae hold for $d=1,2,3,4$ and $\eta \geqslant W^{-d / 3}$. (For $d=4$, power law behaviour is replaced by logarithmic behaviour.) (Universality of mesoscopic eigenvalue statistics)
- For $d \geqslant 5$ : explicit asymptotic computation of $Z\left(E_{1}, E_{2}\right)$. The result is not universal and has no simple expression.
- The constants $K_{d}$ are independent of $\phi$, while $C_{d}$ satisfies

$$
C_{d} \propto \int \mathrm{~d} t|t|^{1-d / 2}|\widehat{\phi}(t)|^{2}
$$

- Wigner matrices (mean-field) satisfy the AS formulae with $d=0$.

Case (b) for Wigner matrices $(d=0)$ was already proved by Boutet de Monvel and Khorunzhy [1999].

Expand $Y(E)$ as a power series in $H$. Main difficulty: terms are highly oscillating.

Need a systematic resummation procedure (perturbative renormalization). We use a two-step renormalization.

1. Instead of expanding in powers $\left\{H^{n}\right\}_{n \in \mathbb{N}}$, expand in Chebyshev polynomials $\left\{U_{n}(H)\right\}_{n \in \mathbb{N}}$.
Idea going back to Bai, Yin, Feldheim, and Sodin. Used in subsequent works of Sodin and Erdős-K.

Classify resulting terms using graphs.
2. Further resum graphs belonging to certain equivalence classes (ladder subdiagram resummation), using explicit identities to exploit strong oscillations.

Requires somewhat involved algebra since estimates are not allowed.
The resulting terms are estimated using pointwise bounds on the resolvent of $S$ (local central limit theorems).

- Diffusion profile with strong control for $L \leqslant W^{5 / 4}$ and $\eta \geqslant W^{-1 / 2}$.
- Eigenvectors are delocalized for $L \leqslant W^{5 / 4}$.
- Proof of the Altshuler-Shklovskii formulae for density-density correlations: mesoscopic universality.

Major open questions:

- Improve $L \leqslant W^{5 / 4}$ to $L \leqslant W^{2}$.
- Control resolvent for $\eta \leqslant W^{-1}$.
- Microscopic universality of random band matrices for $L \leqslant W^{2}$.
(Erdős-K-Yau-Yin [2012]: microscopic universality provided that $L \leqslant W^{34 / 33}$ and $H$ has a vanishing mean-field component.)

