# Universal conductivity properties in many body theory

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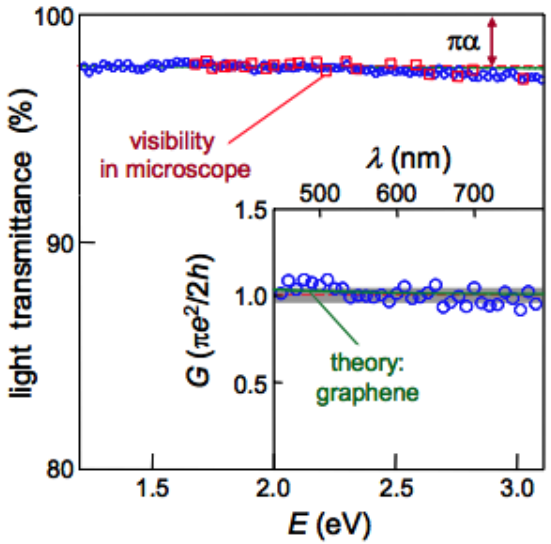
- Several quantum many body systems exhibit some universality properties in the conductivity , which one would like to mathematically derive starting from microscopic models.
- I will consider *optical conductivity* of two physical systems: a) non integrable quantum spin chains b) graphene.
- Solid state models provide concrete realizations of low dimensional QFT models which can be studied by the methods of constructive QFT.

# Universality in the optical conductivity in graphene experiments

Nair Geim et al. Science (2008). The conductivity in a frequency range $\beta^{-1} << \omega << \Lambda$ is $\sigma_0 = \frac{\pi e^2}{2h}$ (universality). N-layer graphene $\sigma_0 = N\frac{\pi e^2}{2h}$ up a few percent.)




They measure the transparency $T$ of light and from that the conductivity $T(\omega) = 1/[(1 + 2\pi\sigma(\omega)]^2$ (in the fig. called $G((\omega))$. Between 2 and 3 eV $\frac{\sigma(\omega)}{\sigma_0} = 1.01 \pm 0.03$

# Non integrable Quantum Spin chains

- The Heisenberg $XXZ$ spin chain $H_0 =$

$$-\sum_{x=1}^{L-1}[JS_x^1 S_{x+1}^1 + JS_x^2 S_{x+1}^2 + J_3 S_x^3 S_{x+1}^3 - hS_x^3]$$

where $S_x^\alpha = \sigma_x^\alpha/2$ for $i = 1, 2, \ldots, L$ and $\alpha = 1, 2, 3$, $\sigma_x^\alpha$ being the Pauli matrices ($J = 1$).

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- The above model can be solved by Bethe ansatz, and it is interesting to add a next-to-nearest neighbor interaction breaking exact solvability, that is consider

$$H = H_0 + H_1$$

$$H_1 = -\lambda\sum_{x=1}^{L-1}[S_x^1S_{x+2}^1 + S_x^2S_{x+2}^2 + S_x^3S_{x+2}^3]$$

# Linear response theory

- By the Peierls substitution $j_x = S_x^1 S_{x+1}^2 - S_x^2 S_{x+1}^1 + \lambda F_x$ where $F_x$ is an expression *quartic* in the spin operators.

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- If $\rho_x = S_x^3 - \frac{1}{2}$ and $(j_x^0, j_x^1) = (\rho_x, j_x)$

$$K_{\beta,\lambda}^{\mu,\nu}(p_0, p) = \lim_{L\to\infty} \int_0^\beta dx_0 e^{-ip_0 x_0} < \hat{j}_{x_0,p}^{\mu} \hat{j}_{x_0,p}^{\nu} >_{\beta,T}$$

and $< O >_\beta = \frac{\mathrm{Tr} e^{-\beta H} \mathcal{T}}{\mathrm{Tr} e^{-\beta H}}$, $O_{x_0} = e^{Hx_0} O e^{-Hx_0}$ (imaginary times), $T$ denotes truncation and $\mathcal{T}$ denotes time ordering.

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- Using the Jordan-Wigner transformation it can be written in terms of fermions $a_x^\pm$.

# Conductivity

- We consider the optical d.c. conductivity (Kubo formula)

$$\sigma_\lambda = \lim_{p_0 \to 0} \lim_{p \to 0} \lim_{\beta \to \infty} \frac{D_{\beta,\lambda}(\mathbf{p})}{i p_0}$$

where $\mathbf{p} = (p_0, p)$ and

$$D_{\beta,\lambda}(\mathbf{p}) = [K^{11}_{\beta,\lambda}(\mathbf{p}) + < j^D >_\beta]$$

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- Give information in the optical regime.
- The zero frequency limit should be take along the real axis but we expect it makes no difference in the optical regime

# Conductivity in the $XXZ$ chain

- In the $XXZ$ chain($J_3 \neq 0, \lambda = 0$), Bethe ansatz provides exact formulas (Yang-Yang '66)

$$D_0 = \frac{\pi}{\bar{\mu}} \frac{\sin \bar{\mu}}{2\mu(\pi - \bar{\mu})}$$

$$\kappa_0 = \frac{\bar{\mu}}{2\pi} \frac{1}{(\pi - \bar{\mu})} \sin \bar{\mu} \qquad v_{s,0} = \frac{\pi}{\bar{\mu}} \sin \bar{\mu}$$

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- If $\lambda \neq 0$ is the conductivity still infinite? Is the universal relation still true?

# Conductivity in the non integrable case

- Benfatto, Falco, Mastropietro Comm. Math.Phys. 2009; PRL 2011; Mastropietro PRE 2013

**Theorem.** *There exists $\varepsilon < 1$ such that, if $|J_3|, |\lambda| \leq \varepsilon$ the zero temperature Drude weight is non vanishing and analytic in $J_3, \lambda$; moreover*

$$D_\lambda = K \frac{v_{s,\lambda}}{\pi} \quad \kappa_\lambda = \frac{K}{\pi v_{s,\lambda}}$$

*with* $K =$

$$1 - \frac{1}{\pi v_{s,\lambda}} [(J_3 + 2\lambda)(1 - \cos 2p_F) + \lambda(1 - \cos 4p_F) + F]$$

*and* $v_s = \sin(p_F) + \tilde{F}$, $\sin p_F = h$ *and* $|F| \leq C\varepsilon^2, |\tilde{F}| \leq C\varepsilon$.

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which was conjectured by Haldane (1980), extending previous ideas by Kadanoff and Wegner (1971) and Luther and Peschel (1975).
- At $\lambda = 0$ it reduces to the Bethe ansatz formulas (but no use of Bethe ansatz is done) $K^{-1} = 2(1 - \frac{\bar{\mu}}{\pi}) =$ $K^{-1} = 1 + \frac{2J_3}{\pi} + O(J_3^2)$ and $v_s = 1 + O(J_3)$

# Conductivity in the non integrable case

- $D_\lambda$ is also connected to the critical exponents by exact relations; for instance if $X$ is the exponent of $< S^3_{\mathbf{x}} S^3_{\mathbf{0}} >$ then

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- Other exponents are determined by $X$ using the Kadanoff relations which can be proven to be true in this model

# Lattice Ward Identites

- By the commutation relations

$$-ip_0 < \hat{\rho}_{\mathbf{p}} \hat{a}^-_{\mathbf{k}} \hat{a}^+_{\mathbf{k}+\mathbf{p}} >_T + p < \hat{j}_{\mathbf{p}} \hat{a}^-_{\mathbf{k}} \hat{a}^+_{\mathbf{k}+\mathbf{p}} >_T =$$
$$[\left\langle \hat{a}^+_{\mathbf{k}} \hat{a}^-_{\mathbf{k}} \right\rangle_T - \left\langle \hat{a}^+_{\mathbf{k}+\mathbf{p}} \hat{a}^-_{\mathbf{k}+\mathbf{p}} \right\rangle_T]$$

$$-ip_0 \hat{K}^{0,0}_{\lambda}(\mathbf{p}) + p\hat{K}^{10}_{\lambda}(\mathbf{p}) = 0$$
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- This implies

$$\hat{K}^{00}_\lambda(p_0, 0) = 0, \quad D_\lambda(0, p) = 0$$

Relation between regularity of the FT of correlations and conductivity; for instance if the FT is continuous the Drude weight is vanishing (what is not not in the case).

# Multiscale analysis

- We perform a multiscale RG analysis and we get that the current-current correlation $K^{0,1}_{\beta,\lambda}(\mathbf{p})$ can be naturally decomposed as sum of two terms where the second contains also the irrelevant terms (Umklapp, non linear bands)

$$K^{1,1}_{\lambda}(\mathbf{x}) = K^{(a)1,1}_{\lambda}(\mathbf{x}) + K^{(b)1,1}_{\lambda}(\mathbf{x})$$

and

$$|K^{(a)1,1}_{\lambda}(\mathbf{x})| \leq \frac{C}{1+|\mathbf{x}|^2}$$

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- Gram bounds (Caianiello (N. Cim. 1956); Gawedzski and Kupiainen (CMP 1985)) and implementing Ward Identities at each RG iteration (vanishing of beta function); (Benfatto Mastropietro CMP 2005 )

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- We need to exploit the idea of emerging symmetries introducing a QFT model describing massless Dirac femions with a momentum regularization and a non local quartic interaction.
- The analysis of the ultraviolet problem is done applying a method applied by Lesniewski (CMP 1987) for the analysis of the Yukava$_2$ model; in the infrared one has to use Ward identites and the asymtotic vanishing of the beta function.

# The effective QFT model

- The effective QFT is expressed directly in terms of Grassmann variables. If $j_\mu = \bar{\psi}_{\mathbf{x}} \gamma_\mu \psi_{\mathbf{x}}$. The partition function is (similar definition for the generating fnction) .

$$\int P(d\psi^{(\leq N)}) e^{\tilde{\lambda}_\infty \int d\mathbf{x} v(\mathbf{x}-\mathbf{y}) j_{\mu,\mathbf{x}} j_{\mu,\mathbf{y}}}$$

where $P(d\psi^{(\leq N)})$, where $\psi = (\psi_1, \psi_{-1})$ have propagator $\chi_N(\mathbf{k}) \frac{\not{\mathbf{k}}}{|\mathbf{k}|^2}$ with a smooth cut-off function vanishing for $|\mathbf{k}| \geq 2^N$ and $v(\mathbf{x}-\mathbf{y})$ a short range symmetric interaction.

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- We can tune by implicit function theorem the bare parameter $\lambda_\infty$ so that the exponents are the same (again the vanishing of beta function is used) and $K_\lambda^{(a)1,1}(\mathbf{x})$ is equal to the correlations of this effective model up to constants and setting $c = v_s$.

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- Of course $\tilde{\lambda}_\infty$ is convergent series in $\lambda$ depending on all the details of the spin hamiltonian.

# Ward Identities for the QFT model

- The advantage of this model is that it verifies an extra chiral symmetry. By performing $\psi_{\mathbf{x},\pm} \longrightarrow e^{i\alpha_{\pm,\mathbf{x}}}\psi_{\mathbf{x},\pm}$, $D_\omega = (-ip_0 + \omega cp)$, $\omega = \pm$, $\rho_\omega = \psi^+_\omega \psi^-_\omega$, in the limit $N \longrightarrow \infty$

$$D_\omega < \hat{\rho}_{\mathbf{p},\omega}\hat{\psi}^+_{\mathbf{k},\omega'}\hat{\psi}^-_{\mathbf{k}+\mathbf{p},\omega'} > +\Delta_N(\mathbf{k},\mathbf{p}) =$$
$$\delta_{\omega,\omega'}[< \hat{\psi}^+_{\mathbf{k},\omega'}\hat{\psi}^-_{\mathbf{k},\omega'} > - < \hat{\psi}^+_{\mathbf{k}+\mathbf{p},\omega'}\hat{\psi}^-_{\mathbf{k}+\mathbf{p},\omega'} >]$$

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- $\Delta_N = \int d\mathbf{k}d\mathbf{p}C_N(\mathbf{k},\mathbf{p}) < \hat{\psi}^+_{\mathbf{k}',\omega}\hat{\psi}^-_{\mathbf{k}'+\mathbf{p},\omega}\hat{\psi}^+_{\mathbf{k},\omega'}\hat{\psi}^-_{\mathbf{k}+\mathbf{p},\omega'} >$ with

$$C_N(\mathbf{k},\mathbf{p}) = [(\chi_N^{-1}(\mathbf{k}+\mathbf{p})-1)D_\omega(\mathbf{k}+\mathbf{p}) - (\chi_N^{-1}(\mathbf{k})-1)D_\omega(\mathbf{k})]$$

# Anomalies

- The correction term is non vanishing in the $N \to \infty$ limit. By a multiscale analysis it is found, in the limit of removed cut-off (Mastropietro JMP 2007)

$$\lim_{N\to\infty} \Delta_N(\mathbf{k},\mathbf{p}) = \tau < \hat{\rho}_{\mathbf{p},-\omega} \hat{\psi}^{+}_{\mathbf{k},\omega'} \hat{\psi}^{-}_{\mathbf{k}+\mathbf{p},\omega'} >$$

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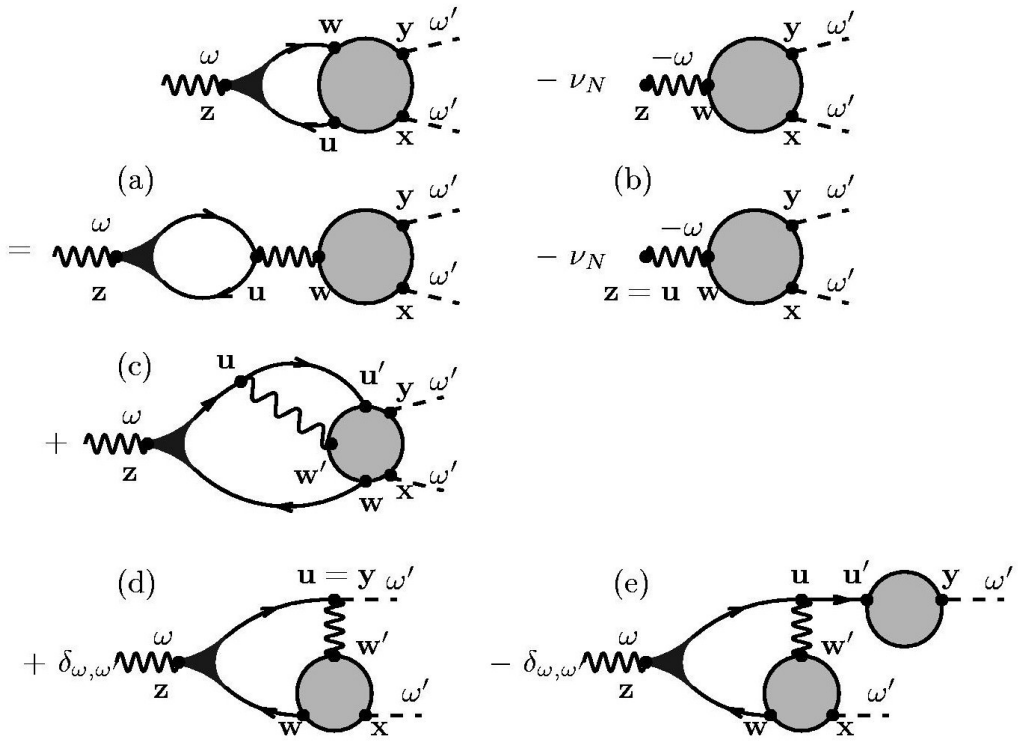
- The coefficient $\tau$ is linear in $\tilde{\lambda}_\infty$: in the case of the axial WI, this is the non-perturbative analogue of the anomaly non renormalization in QED4 (the above WI were postulated by Johnson (Nuovo Cim. 1961) using a self-consistence argument).




Multiscale analysis for $\Delta_N(\mathbf{k}, \mathbf{p})$; decomposition of the new terms with terms with marginal dimension, $\nu_N = \frac{\tilde{\lambda}_\infty}{4\pi c}$.

# New relations for the lattice model

- Similarly we get two Ward Identites for the densities for which an exact expression for them is obtained

$$\hat{K}_{\lambda}^{(a)1,1}(\mathbf{p}) = \frac{1}{4\pi v_s Z^2}\frac{(\tilde{Z}^{(1)})^2}{1-\tau^2}[\frac{D_-(\mathbf{p})}{D_+(\mathbf{p})} + \frac{D_+(\mathbf{p})}{D_-(\mathbf{p})} + 2\tau]$$

$$\hat{K}_{\lambda}^{(a)0,0}(\mathbf{p}) = \frac{1}{4\pi v_s Z^2}\frac{(\tilde{Z}^{(0)})^2}{1-\tau^2}[\frac{D_-(\mathbf{p})}{D_+(\mathbf{p})} + \frac{D_+(\mathbf{p})}{D_-(\mathbf{p})} + 2\tau]$$

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  where $\tau = \frac{\lambda_\infty}{4\pi v_s}$, $D_\omega(\mathbf{p}) = -ip_0 + \omega v_s p$. In order to get that it is essential that we can study both models via multiscale analysis.
- $\tilde{Z}^{(0)} \neq \tilde{Z}^{(1)}$ as irrelevant terms breaks Lorentz symmetry; $v_s, Z, \lambda_\infty, \tilde{Z}^{(0)}, \tilde{Z}^{(1)}$ depend from all microscopic details.

# Sketch of the proof

- On the other hand the parameters are not all independent; the condition $D_\lambda(0,p) = 0$ fixes the value of $\hat{K}_\lambda^{(b)1,1}(0)$.

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- From the WI of the effective model

$$\tilde{Z}[-ip_0\frac{1}{\tilde{Z}^{(0)}}\hat{<}\hat{\rho}_{\mathbf{p}}\hat{a}^+_{\mathbf{k}}\hat{a}^-_{\mathbf{k}+\mathbf{p}}> +pv_s\frac{1}{\tilde{Z}^{(1)}}<\hat{j}_{\mathbf{p}}\hat{a}^+_{\mathbf{k}}\hat{a}^-_{\mathbf{k}+\mathbf{p}}>]=$$

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# Sketch of the proof

- On the other hand the parameters are not all independent; the condition $D_\lambda(0,p)=0$ fixes the value of $\hat{K}_\lambda^{(b)1,1}(0)$.
- From the WI of the effective model

$$\tilde{Z}[-ip_0\frac{1}{\tilde{Z}^{(0)}}<\hat{\rho}_{\mathbf{p}}\hat{a}^+_{\mathbf{k}}\hat{a}^-_{\mathbf{k}+\mathbf{p}}>+pv_s\frac{1}{\tilde{Z}^{(1)}}<\hat{j}_{\mathbf{p}}\hat{a}^+_{\mathbf{k}}\hat{a}^-_{\mathbf{k}+\mathbf{p}}>]=$$
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- The bare parameters are not independent but fixed by the lattice WI

$$\frac{1}{1-\tau}\frac{\tilde{Z}^{(0)}}{\tilde{Z}}=1 \quad \frac{v_s\tilde{Z}^{(0)}}{\tilde{Z}^{(1)}}=1$$

# Sketch of the proof

- In conclusion

$$\hat{K}^{00}_{\lambda}(\mathbf{p}) = \frac{K}{\pi v_s} \frac{v_s^2 p^2}{p_0^2 + v_s^2 p^2} + O(\mathbf{p})$$

$$\hat{D}_{\lambda}(\mathbf{p}) = \frac{K v_s}{\pi} \frac{p_0^2}{p_0^2 + v_s^2 p^2} + O(\mathbf{p})$$

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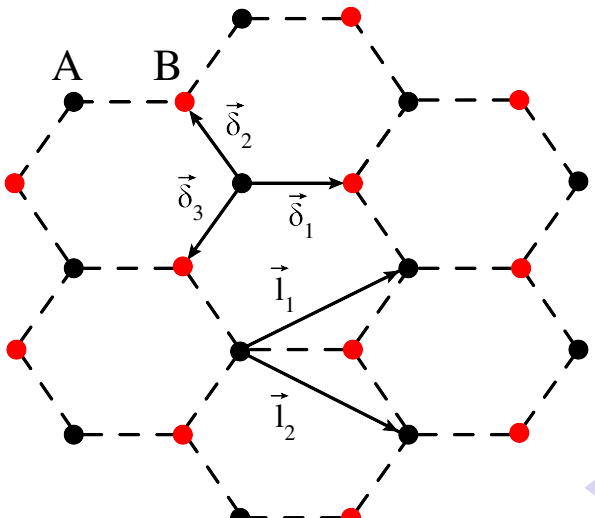
- Crucial role of the irrelevant terms; in field theoretic RG they are neglected, but the problem is exactly to show that universality persists even taking them into account.
- Extension to spinning 1D fermions (Hubbard model) done in Benfatto-Falco-Mastropietro (2013); higly non trivial due to log-divergences modulating the power law.
- Hopefully an extension of the RG analysis at $\beta < \infty$ is possible (Big debate on $D_\beta$).

# Hubbard model on the honeycomb lattice

$$H_U = -t \sum_{\vec{x}\in\Lambda, i=1,2,3} \sum_{\sigma=\uparrow\downarrow} \left( a^+_{\vec{x},\sigma} b^-_{\vec{x}+\vec{\delta}_i,\sigma} + b^+_{\vec{x}+\vec{\delta}_i,\sigma} a^-_{\vec{x},\sigma} \right) +$$

$$U \sum_{\substack{\vec{x}\in\Lambda \\ i=1,2,3}} \sum_{\sigma,\sigma'} \left( a^+_{\vec{x},\sigma} a^-_{\vec{x},\sigma} - \frac{1}{2} \right) \left( b^+_{\vec{x}+\vec{\delta}_i,\sigma'} b^-_{\vec{x}+\vec{\delta}_i,\sigma'} - \frac{1}{2} \right)$$

$a^\pm_{\vec{x}}, b^\pm_{\vec{x}}$ fermionic operators,
$\vec{\delta}_1 = (1,0)\ , \quad \vec{\delta}_2 = \frac{1}{2}(-1,\sqrt{3})\ , \quad \vec{\delta}_3 = \frac{1}{2}(-1,-\sqrt{3})$, $\Lambda \equiv \Lambda_A$
periodic triangular lattice



# Physical observables

1. $\Psi^{\pm}_{\vec{x},\sigma} = \left(a^{\pm}_{\vec{x},\sigma}, b^{\pm}_{\vec{x}+\vec{\delta}_1,\sigma}\right)$, $\Psi^{\pm}_{\mathbf{x},\sigma} = e^{Hx_0}\Psi^{\pm}_{\vec{x},\sigma}e^{-Hx_0}$ with $\mathbf{x} = (x_0, \vec{x})$ and $x_0 \in [0, \beta]$, for some $\beta > 0$.

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2. If $S(\mathbf{x} - \mathbf{y}) = \langle \Psi^{-}_{\mathbf{x}} \Psi^{+}_{\mathbf{y}} \rangle_{\beta}$ we denote by $\hat{S}(\mathbf{k})$ the F.T., $\mathbf{k} = (k_0, \vec{k})$, $k_0 = \frac{2\pi}{\beta}(n_0 + \frac{1}{2})$ : $n_0 \in \mathbb{Z}$, $\vec{k} \in \mathcal{B}$ the first Brillouin zone.

# The 2-point function for $U = 0$

1.

$$S_0(\mathbf{k}) = \frac{1}{k_0^2 + |v_F^{(0)}\Omega(\vec{k})|^2} \begin{pmatrix} ik_0 & -v_F^{(0)}\Omega^*(\vec{k}) \\ -v_F^{(0)}\Omega(\vec{k}) & ik_0 \end{pmatrix},$$

$$v_F^{(0)}\Omega(\vec{k}) = t\sum_{i=1}^{3} e^{i\vec{k}(\vec{\delta}_i - \vec{\delta}_1)} = t(1 + 2e^{-i3/2k_1}\cos\frac{\sqrt{3}}{2}k_2).$$

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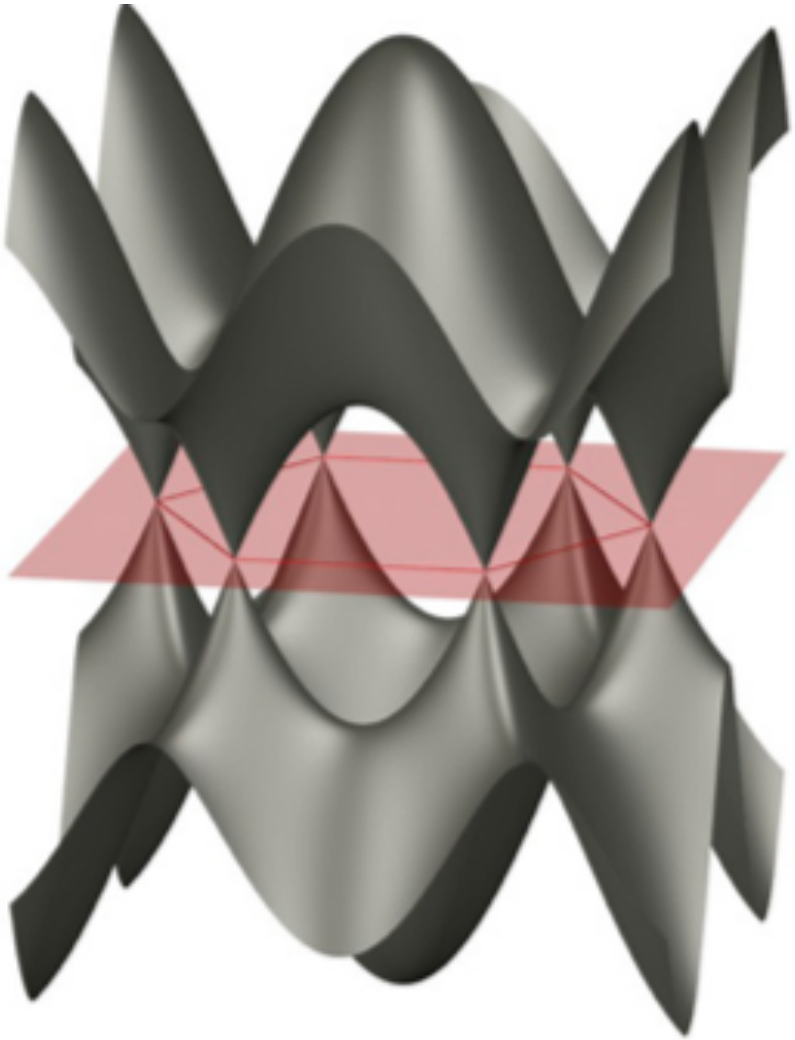
2. If $\vec{p}_F^{\,\pm} = (\frac{2\pi}{3}, \pm\frac{2\pi}{3\sqrt{3}})$, $v_F^{(0)} = \frac{3}{2}t$ close to a Dirac propagator (massless Dirac in $2+1$ while in the previous case $1+1$)

$$S_0(\mathbf{k} + \mathbf{p}_F^{\pm}) \sim \begin{pmatrix} ik_0 & v_F^{(0)}(ik_1' \mp k_2') \\ v_F^{(0)}(-ik_1' \mp k_2') & ik_0 \end{pmatrix}^{-1},$$

# The dispersion relation



# The optical conductivity

- The currents are (spin is understood)

$$\vec{\hat{J}}_{\vec{p}} = iet \sum_{\substack{\vec{x}\in\Lambda \\ ,j}} e^{-i\vec{p}\vec{x}} \vec{\delta}_j \eta^j_{\vec{p}} \left( a^+_{\vec{x}} b^-_{\vec{x}+\vec{\delta}_j} - b^+_{\vec{x}+\vec{\delta}_j} a^-_{\vec{x}} \right) = v_F^{(0)} \vec{\hat{\hat{J}}}_{\vec{p}}$$

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- The conductivity at imaginary frequencies by Kubo formula is $\omega = \frac{2\pi}{\beta} n$

$$\sigma^\beta_{lm}(i\omega) = -\frac{2}{3\sqrt{3}} \frac{e^2}{\hbar\omega} \left[ (v_F^{(0)})^2 < \hat{j}_{l,\omega,0}; \hat{j}_{m,-\omega,0} >_\beta + \Delta^\beta_{lm} \right],$$

where $3\sqrt{3}/2$ is the area of the hexagonal cell, $< \hat{j}_{l,\omega,\vec{p}}; \hat{j}_{m,-\omega,\vec{p}} >= FT(< \hat{j}_{l,x_0,\vec{p}}; \hat{j}_{m,y_0,-\vec{p}} >)$.

# The optical conductivity for $U = 0$: theoretical predictions

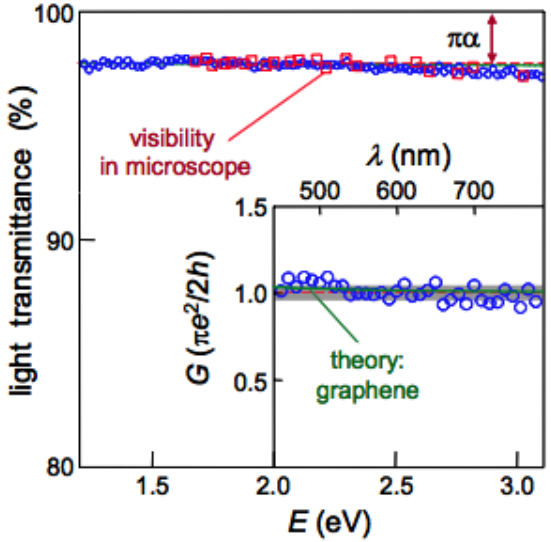
- Stauber, Peres, Geim PRB (2008)

$$\lim_{\omega \to 0} \lim_{\beta \to \infty} \sigma^{\beta}_{lm}(\omega + i0^+) = \delta_{lm}\sigma_0 \qquad \sigma_0 = \frac{\pi e^2}{2h}$$

Universal conductivity ($t$ independent) for $\omega$ small and greater than $\beta^{-1}$. *Finite* as the density of states is vanishing.

# The optical conductivity: experiments

Nair et al. Science (2008). The conductivity in a frequency range $\beta^{-1} << \omega << t$ is $\sigma_0 = \frac{\pi e^2}{2h}$ (universality) up a few percent (In the same range the conductivity for N-layer graphene is $\sigma_0 = N\frac{\pi e^2}{2h}$ up a few percent.)




They measure the transparency $T$ of light and from that the conductivity $T(\omega) = 1/[(1 + 2\pi\sigma(\omega)]^2$ (in the fig. called $G((\omega))$. Between 2 and 3 eV $\frac{\sigma(\omega)}{\sigma_0} = 1.01 \pm 0.03$

# Experiments and some puzzle

- The electron-electron interaction is large $e^2/\hbar v_F^0 \sim 2.18$
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- The electron-electron interaction is large $e^2/\hbar v_F^0 \sim 2.18$ Why the conductivity is universal, that is there is no an essential many body renormalization in the conductivity?
- Exacerbating the problem, in other experiments the interaction appear. Ellis et al Nat. Mat. (2011): the Fermi velocity is strongly enlarged by the interactions at low frequencies.
- There is a large debate in current times on the graphene conductivity. In particular some people have found interaction dependent corrections while others objects that these are spurious effects due to the uv regularizations.

# Universality of the conductivity

- Giuliani, Mastropietro. CMP 293,301 (2010); PRB(R)79, 201403 (2009); Giuliani, Mastropietro, Porta. PRB 83, 195401 (2011); CMP 311,317 (2012).

**Theorem**

*For $|U| \leq U_0$ and any fixed $\omega$, $\sigma^{\beta}_{lm}(i\omega)$ is analytic in $U$ uniformly in $\beta$ and*

$$\lim_{\omega \to 0^+} \lim_{\beta \to \infty} \sigma_{lm}(i\omega) = \frac{e^2}{h} \frac{\pi}{2} \delta_{lm} \, .$$

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- While the Fermi velocity and the wave function renormalization are renormalized $v_F(U) > v_F(0)$ the conductivity is protected: radiative corrections cancel out.

# Proof.

- The correlation is then written as a convergent (due to Gram bounds) tree expansion at weak coupling and , if $\hat{K}_{lm}(\mathbf{p})$ is the FT of $\langle J_{l,\mathbf{x}}; J_{m,\mathbf{y}} \rangle$ and $\hat{K}_{0m}(\mathbf{p})$ is the FT of $\langle \rho_{\mathbf{x}}; J_{m,\mathbf{y}} \rangle$, from the bound

$$|K_{\mu,\nu}(\mathbf{x})| \leq \frac{C}{1 + |\mathbf{x}|^4} \ ,$$

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  $\hat{K}_{\mu\nu}(\mathbf{p})$ is continuous at $\mathbf{p} = \mathbf{0}$
- Now the WI implies that the Drude weight is vanishing

$$\sigma_{lm} = -\frac{2}{3\sqrt{3}} \lim_{\omega\to 0^+} \lim_{\beta\to\infty} \frac{1}{\omega}\left[\hat{K}_{lm}(\omega,\vec{0}) - \hat{K}_{lm}(\mathbf{0})\right] .$$

  $K_{l,m}(\mathbf{p})$ is even: if the derivative were continuous the conductivity vanishes. But is not. (CFR $1D$ $\hat{K}_{l,m}$ non continuous $\sigma(0) = \infty$)

# The current-current function

- As a result of the Renormalization Group analysis and tree expansion

$$\hat{K}_{lm}(\mathbf{p}) = \frac{Z_l Z_m}{Z^2} \langle \hat{j}_{\mathbf{p},l}; \hat{j}_{-\mathbf{p},m} \rangle_{0,v_F} + \hat{R}_{lm}(\mathbf{p})$$

where $\langle \cdot \rangle_{0,v_F}$ is the average associated to a non-interacting system with Fermi velocity

$$v_F(U) = \frac{3}{2}t + dU + .. \quad Z_\mu = 1 + aU + bU^2 + ..$$

and

$$|R_{lm}(\mathbf{x},\mathbf{y})| \leq \frac{C}{1 + |\mathbf{x} - \mathbf{y}|^{4+\theta}}$$

with $0 < \theta < 1$ (power counting improvement due to irrelevance), so that $\hat{R}_{lm}(\omega, \vec{0})$ is continuous and differentiable at $\mathbf{p} = \mathbf{0}$.

# Implications of WI

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3. Note that $\hat{K}_{lm}(\mathbf{p})$ is even

# Universality of the conductivity

- Finally

$$\sigma_{11} = -\frac{2}{3\sqrt{3}} \lim_{\omega \to 0^+} \frac{1}{\omega} \Big[ \big( \hat{R}_{11}(\omega, \vec{0}) - \hat{R}_{lm}(0, \vec{0}) \big) + \big( v_F^2 \langle \hat{j}_{(\omega,\vec{0}),l} ; \hat{j}_{(-\omega,\vec{0}),m} \rangle_{0,v_F} - v_F^2 \langle \hat{j}_{\mathbf{0},l} ; \hat{j}_{\mathbf{0},m} \rangle_{0,v_F} \big) \Big] .$$

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- The first term is differentiable and even hence vanishing, while the first term is identical to the free one so it does not depend from $v_F$ ■

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- This is consequence of the Fermi velocity divergence, a rather unphysical phenomenon.
- However if we take into account retardation effects, there is emergence of Lorentz symmetry and the Fermi velocity flows to the light velocity (Giuliani Mastropietro Porta Ann. Phys. 2012).
- In this case the conductivity is different from the non interacting one, but still universal (Herbut-Mastropietro PRB 2013) does not depend from the material parameter.[Results order by order in the perturbative expanasion]

# Conclusion

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- Rigorous RG methods allow the proof of several universality properties.
- Lattice effects are important even if they are irrelevant in the RG sense.
- (Non trivial) extensions would include finite temperature effects (role of integrability) and disorder.