The Role of Symmetry in Statistical Mechanics and Random Matrices

> Tom Spencer Institute for Advanced Study Princeton, NJ

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Spin systems on \mathbb{Z}^d with continuous symmetry .

Motivation: Spectral Theory of Random Band Matrices **Examples:** $j \in \mathbb{Z}^d \cap \Lambda$, $\Lambda = Large$ Cube of side L

 $s_j \in Unit circle, X - Y model, O(2)$

 $s_{j} \in Unit Sphere, Classical Heisenberg, O(3)$ $s_{j} \in H^{2}, Hyperbolic plane = SU(1,1)/U(1)$ $s_{j} \in H^{(2|2)} = SUSY$ Hyperbolic sigma model $s_{i} \in U(1,1|2)/[U(1|1) \times U(1|1)] = Efetov Sigma model.$

To Describe Ordered Phase at Low Temperature - 3D

Outline of Lecture .

A) Review of X-Y and Heisenberg models

B) **Conjecture:** Universality of Mean Field Theory Symmetry governs the low temperature phase - 3D

Wigner-Dyson Universality of local eigenvalue correlations Kravstov-Mirlin ('94) finite volume corrections to W-D.

C) Gaussian Random Band Matrices: H_{jk} , $j, k \in \Lambda \cap \mathbb{Z}^d$.

$${\sf Z}_{\Lambda}(eta)pprox < \det(H_{\Lambda}-E)^2>$$

 $\beta = W^2 \rho(E)^2$, $H_{jk} \approx 0$ for $|j - k| \gg W$ = band width. $\mathbf{Z}_{\Lambda}(\beta)$ = partition function for Heisenberg Model. D) SUSY $H^{2|2}$ model - due to Zirnbauer ('91)

DSZ (10): "Anderson-like" transition in 3D. - Localization-Diffusion.

Sabot and Tarres $H^{2|2}$ equivalent to: **VRJP** Vertex Reinforced Jump Process

Edge Reinforced Random Walk equivalent to $\tilde{H}^{2|2}$

Speculations about exponential localization in 2D.

History dependent walks and SUSY

Linearly edge reinforced random walk (ERRW), is a discrete time walk on \mathbb{Z}^d starting at the origin. Let **n(e, t)** denote the number of times the walk has visited the edge *e* up to time t.

The probability P(j, j'; t + 1) walk at vertex j will visit a neighboring edge (j, j') at time t + 1 is given by

$$egin{aligned} & P(j,j';t+1) = [1+\mathsf{n}(\mathbf{j},\mathbf{j}';\mathbf{t})/eta]/S_eta(j,t) \ & S_eta = \sum_{e,\,j\in e} [1+\mathsf{n}(e,\mathbf{t})/eta] \end{aligned}$$

P. Diaconis: Partially exchangeable process \approx **RWRE VRJP** is similar but depends on local time at vertices.

X-Y and Heisenberg models on $\Lambda_L \cap \mathbb{Z}^d$

$$Z_{\Lambda}(\beta) \equiv \int e^{\beta \sum_{j \sim j'}^{\Lambda} s_{j} \cdot s_{j'}} \prod_{j \in \Lambda} d\mu(s_{j})$$

 $d\mu(s)$ is Haar measure on **S**¹ or **S**², Λ = periodic box.

$$\langle A \rangle_{\Lambda}(\beta) \equiv rac{1}{Z_{\Lambda}(\beta)} \int A e^{\beta \sum_{j \sim j'}^{\Lambda} s_{j} \cdot s_{j'}} \prod_{j \in \Lambda} d\mu(s_j)$$

$$M^2_{\Lambda}(eta) \equiv |\Lambda|^{-2} \sum_{j,k \in \Lambda} \langle s_j \cdot s_k \rangle_{\Lambda}(eta)$$

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Order in 3 dimensions, $\beta \gg 1$, [FSS ('76)],

$$M^2_{\Lambda}(\beta) o M^2(\beta) pprox 1 - rac{n-1}{eta} G(0,0) \quad \text{as } L o \infty \,.$$

where G is the Green's function for $-\Delta$; n=2, n=3

Proofs: Reflection Positivity, RNG (Balaban), Fourier Analysis + Ginibre for O(2) case

Order for 3D SUSY $H^{2|2}$ [DSZ].

Proof: SUSY Ward Identities $Z(\beta) \equiv 1$, estimates for non-uniformly elliptic Random Walk. Induction on length scales.

Mermin-Wagner: In 2D, $M_{\Lambda}(\beta) \rightarrow 0$ as $L \rightarrow \infty$

The 2D X-Y model has *power law* decay for $\beta \gg 1$:

$$[Fr - Sp]$$
: $\langle s_0 \cdot s_x \rangle(\beta) \approx \frac{1}{|x|^{1/(2\pi\bar{\beta})}}$

Conjecture: At β_c , $\langle s_0 \cdot s_x \rangle (\beta_c) \approx \frac{1}{|x|^{1/4}}$, KT transition.

Remarks: The 2D X-Y model is dual to the 2D Coulomb gas: vortices \approx charges.

Falco '11 analyzes the dilute 2D Coulomb gas at β_c - RNG.

Polyakov Conjecture for 2D Heisenberg:

$$\langle s_0 \cdot s_j
angle(eta) pprox e^{-|\mathbf{x}|/\ell(eta)}, \quad \ell(eta) pprox e^{Ceta} \gg 1$$

Conjecture: The 2D $H^{2|2}$, and Edge Reinforced Random Walk are **exponentially localized**.

Pin $c_{jj'} = 1$ for j = 0, where $c_{jj'} = local$ conductance $j \sim j'$ For 2D ERRRW, **MerkI-Rolles:**

$$0 \leq < c_{jj'}^{1/4} > \leq C|j|^{-1/c\beta}$$
.

Exponential decay?

3D Mean Field Conjecture

Let $\xi \in \mathbb{R}^n$, and $\Lambda_L \subset \mathbb{Z}^3$, periodic cube side $L \to \infty$

$$\langle e^{\frac{1}{|\Lambda|}\sum_{j\in\Lambda}\xi\cdot s_{j}}\rangle(\beta) \to \int_{\mathbf{S}^{n-1}} e^{M_{L}(\beta)\xi\cdot s_{0}}d\mu(s_{0})$$
Correction: $\times \exp\left[\frac{C|\xi|^{2}}{\overline{\beta}^{2}L^{2(d-2)}}\right]$ - Free Field

Interpretation: Law of Large Numbers with CLT correction.

D. Ueltchi: Related conjecture for Quantum Heisenberg: Poisson-Dirichlet process - coagulation - fragmentation of long permutation cycles.

Remarks

- 1) Valid to second order in ξ by definition.
- 2) Formal derivation for SUSY Efetov model Kravtsov-Mirlin Spin Wave analysis about ordered state. Wigner - Dyson

M $\approx \rho(E)$ and $\bar{\beta}L^{(d-2)} =$ conductance.

- 3) Valid for X-Y for almost all β [Fröhlich, Fr-Pfister] with Correction $\approx 1/\beta L^{(d-2)}$
- 4) Conjecture does not depend on sigma approximation ϕ^4 OK.

Sketch of proof [Fröhlich]:

For the X-Y model, the infinite volume state: $\langle \cdot \rangle$

$$\langle exp\{\frac{1}{|\Lambda|}\sum_{j\in\Lambda}\xi\cdot s_j\}\rangle(\beta) = \int \langle exp\{\frac{1}{|\Lambda|}\sum_{j\in\Lambda}\xi\cdot s_j\}\rangle(\beta, s_0)d\mu(s_0)$$

where s_0 labels the pure states. Note

$$\langle \xi \cdot s_j \rangle (\beta), s_0) = M \xi \cdot s_0$$

Correction: $Var_{s_0}\{L^{-d}\sum_j s_j \cdot \xi\} \leq \frac{c}{\beta L^{(d-2)}}$ - IR bounds.

Gaussian Random Band Matrices

Let $H = H^*$ be a random matrix indexed by $j, k \in \Lambda_L \cap \mathbb{Z}^d$ such that H_{jk} is Gaussian, $\langle H_{jk} \rangle = 0$ and

$$\langle H_{jk}\bar{H}_{j'k'}\rangle = \delta_{j,j'}\delta_{k,k'}J_W(j,k) \quad j,k\in\Lambda_L\cap\mathbb{Z}^d$$

and

$$egin{aligned} & J_W(j,k) \equiv (-W^2 \Delta_\Lambda + 1)^{-1}(j,k) \ & pprox W^{-1} e^{-|j-k|/W} & ext{in } 1D \end{aligned}$$

W is the width of the band.

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The Green's Function, $E_{\epsilon} = E + i\epsilon$

$$G(E_{\epsilon};j,k) \equiv (\mathbf{H}-E_{\epsilon})^{-1}(j,k).$$

Extended States, Quantum-Diffusion in 3D:

$$< |G(E_{\epsilon}; j, k)|^2 > \cong rac{
ho(E)}{-D\Delta + \epsilon} (j, k) \approx C (|j - k| + 1)^{-1}$$

for $j, k \mathbb{Z}^3$ and $|E| \le 1.9$, $ho(E)$ = density of states.

Localization :

$$< |G(E_{\epsilon};j,k)|^2 > \cong \epsilon^{-1} e^{-|j-k|/\ell(E)|}$$

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Conjecture: In 1D, if $W^2 \gg L$,

Then local pair correlation for RBM = Mean Field GUE.

Theorem [T. Shcherbina]: 1D case If $W^2 \gg L$, and -2 < E < 2, then as $L \to \infty$ $/det(H - E + \xi/L\pi a) det(H - E - \xi/L\pi a)$

$$R \equiv \frac{\langle \det(H - E + \xi/L\pi\rho) \det(H - E - \xi/L\pi\rho) \rangle}{\langle \det(H - E)^2 \rangle}$$
$$\rightarrow \int_{\mathbf{S}^2} e^{i\xi \cdot s_0} d\mu(s_0) = \frac{\sin|\xi|}{\xi}$$

where $\rho = \rho(E)$

Conjecture: In 3D Result holds for *fixed* large W. **Remark:** There are simple corrections for non Gaussian case.

Sketch of proof:

Let $\psi, \bar{\psi}, and \chi, \bar{\chi}$ be anti commuting Grassmann variables.

$$Det(H-E) = \int e^{-\sum_{jk\in\Lambda} \bar{\psi}_j (H_{jk}-E\delta_{jk})\psi_j} D\psi$$

If H is Gaussian band matrix we can intergrate over H to get

$$\langle Det(H-E)^2
angle = \int e^{-\sum_{jk} [J_{jk} tr M_j M_k - EM_j \delta_{jk}]} D\psi D\chi$$

where $M_j = \begin{pmatrix} \bar{\psi}_j \psi_j & \bar{\psi}_j \chi_j \\ \bar{\chi}_j \psi_j & \bar{\chi}_j \chi_j \end{pmatrix}$.

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Let X_i be 2 × 2 hermitian matrices - dual to M.

then by Hubbard Stratonovich we have

$$\langle Det(H-E)^2 \rangle$$

$$=\int e^{-\frac{1}{2}\sum_{j}^{\Lambda}[W^2tr(\nabla X)_j^2+trX_j^2]}\prod_j det(X_j-iE)dX_j$$

For large W saddle point dominates:

$$X_j \approx U_j^* \operatorname{diag}(
ho(E), -
ho(E)) U_j \quad U \in SU(2)$$

In this approximation we have a Heisenberg model with

$$X_j =
ho(E)s_j, \quad s_j \in \mathbf{S}^2, \quad \beta = W^2
ho(E)^2$$

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SUSY Statistical Mechanics - Efetov

Averages of Green's functions can be rigorously written as correlations of **SUSY** Statistical Mechanics - **duality**.

The spins in SUSY models are 4 by 4 matrices Φ_j which have 8 real components and 8 anticommuting components.

The action $A(\Phi)$ has SU(1,1|2) invariance -

Hyperbolic and Heisenberg models are coupled.

Advantages: Randomness can be integrated out. A finite dimensional **Saddle manifold** dominates the integral. Nonperturbative analysis, Symmetries manifest.

Efetov SUSY sigma model - One Dimension

Let $h_j \in H^2$ and $\sigma_j \in S^2$ be spins in hyperboloid and sphere. The Gibbs weight is proportional to

$$\prod_{j=0}^{L} (h_j \cdot h_{j+1} + \sigma_j \cdot \sigma_{j+1}) e^{\beta(\sigma_j \cdot \sigma_{j+1} - h_j \cdot h_{j+1})}.$$

Where
$$h_j = (x_j, y_j, z_j)$$
 satisfy $z_j^2 - x_j^2 - y_j^2 = 1$ and $h \cdot h' = zz' - xx' - yy'$.

It is convenient to parameterize this hyperboloid with **horospherical** coordinates $s, t \in \mathbb{R}$:

$$z = \cosh t + s^2 e^t / 2$$
, $y = \sinh t - s^2 e^t / 2$, $x = s e^t / 2$

Theorem [Dis-S] Conductance $\approx e^{-L/\beta}$, $\beta = \rho(E)^2 W^2$.

$H^{2|2}$ supersymmetric hyperbolic model on \mathbb{Z}^d

Simplified model of Anderson localization - delocalization. The spins lie on a hyperboloid with Grassmann components:

$$m{s}_j = (z_j, x_j, y_j, ar{\psi}_j, \psi_j)$$
 with $m{s}_j \cdot m{s}_j = 1$
 $m{s} \cdot m{s}' \equiv zz' - xx' - yy' - 2ar{\psi}\psi'$

Nearest Neighbor interaction with small magnetic field $\epsilon > 0$

$$e^{-\beta \sum_{j,j'}^{\Lambda} s_j \cdot s_{j'}} e^{-\epsilon \sum^{\Lambda} z_j}$$

Correlation: Note ($\epsilon \approx$ imaginary part of energy)

$$\mathcal{G}_{\epsilon}(j,k) \equiv \langle y_j y_k \rangle (\beta,\epsilon)$$

Localization - Diffusion transition for $H^{2|2}$ in 3D

Disertori, Sp, Zirnbauer (2010): In infinite volume limit For $\beta \gg 1$ diffusion in 3D,

$$\mathcal{G}_{\epsilon}(j,k) pprox (-D\Delta + \epsilon)^{-1}(j,k)$$

For $0 < \beta \ll 1$, localization:

$$\mathcal{G}_{\epsilon}(j,k) \leq rac{C}{\epsilon} e^{-|j-k|/\ell}$$

uniformly as $\epsilon \downarrow 0$

 $H^{2|2}$ as a Random Walk in Random Environment Let $t_j \in \mathbb{R}, \ j \in \Lambda \cap \mathbb{Z}^d$. Define elliptic operator **D**

$$[\mathbf{v};\mathbf{D}_{\beta,\varepsilon}(t)\mathbf{v}]_{\Lambda} \equiv \beta \sum_{(j\sim j')} \mathrm{e}^{t_j+t_{j'}} (\mathbf{v}_j - \mathbf{v}_{j'})^2 + \sum_{k\in\Lambda} \varepsilon_k \,\mathrm{e}^{t_k} \mathbf{v}_k^2$$

The local conductance $= e^{t_j + t_{j'}}$ for $j \sim j'$.

After integrating out the Grassmann variables, distribution of random environment $\{t_j\}$ has Gibbs density:

$$e^{-eta \mathcal{L}(t)} \cdot [\det D_{eta,arepsilon}(t)]^{1/2} \prod_j e^{-t_j} rac{dt_j}{\sqrt{2}\pi}$$

where

$$\mathcal{L}(t) = \sum_{j \sim j'} [\mathit{cosh}(t_j - t_{j'}) - 1] + \sum_j rac{arepsilon_j}{eta} [(\mathsf{cosh}(t_j) - 1] .$$

Behavior of local conductance

In 3D, for $\beta \gg 1$, moments of the conductance are bounded

$$\langle e^{p(t_j+t_{j'})} \rangle (\beta,\epsilon) \leq C \quad p=\pm 2,\pm 4$$

uniformly in ϵ . - Ordered, **Diffusive phase**

Localization follows from exponential decay of the conductance

$$\langle e^{(t_j+t_{j'})/4} \rangle \leq e^{-|j|/\ell}$$
 for $|j| \gg 1$.

Merkl-Rolles: In 1D strip, $\ell \approx$ width of strip, In 2D power law decay - ERRW. Mermin Wagner.

Conjecture: In 2 Dimensions localization holds for all $\beta < \infty$ for ERRW and $H^{2|2}$.

Control t fluctuations via SUSY Ward identities

For Example for any $\ell > 1$,

$$< cosh^m(t_0-t_\ell)(1-rac{m}{eta}G_\ell(t))>=1$$

where G_{ℓ} is the Green's function of $\mathbf{D}(t)$:

$$egin{aligned} G_\ell(t) &pprox e^{t_0+t_\ell} \left((\delta_0-\delta_\ell), \{ \mathbf{D}_{eta,\epsilon}(t) \}^{-1} (\delta_0-\delta_\ell)
ight) \ & ext{If } 0 \leq G_\ell(t) \leq \mathbf{C}, ext{ then for } \mathbf{C}m < eta \ &< cosh^m(t_0-t_\ell) > \leq (1-rac{\mathbf{C}m}{eta})^{-1} \end{aligned}$$

Implies that t - fluctuations at scale ℓ are small for large β .

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2D localization for $H^{2|2}$??

A) ERRW and VRJP are attractive. 2D RW is recurrent so attractiveness is enhanced.

B) The 2D saddle point (minimum) in t of $H^{2|2}$ exhibits localization - conductance goes to 0 exponentially fast.

C) The Hessian in at $t \approx 0$ is given by:

$$-\beta \sum_{j} (\nabla t_j)^2 + \sum_{j,k} (\nabla t_j) J(j-k) (\nabla t_k)$$

where

$$J(j-k) = [\nabla G_0(j,k)]^2 \approx \frac{1}{|j-k|^{2(d-1)}}$$

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Log divergent sum in 2D ! effective $\beta \rightarrow 0$ in 2D.

Comments

A) Need more robust techniques to prove and analyze continuous symmetry breaking. Simple O(2) models should be analyzed in more detail.

B) Can ERRW or VRJP analyzed directly without SUSY?

Direct probabilistic methods may offer insights into 2D localization and the Anderson phase transition in 3D.

At present there there is little understanding of transition for ERRW or $H^{2|2}$ in 3D. Probably no upper critical dimension and Multi-fractal.