The first-order transition of the ground-state in the quantum random energy model

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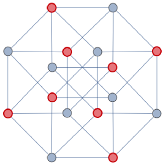
1. The Quantum Random Energy Model



Hamming cube: $Q_N := \{-1,1\}^N$

- configuration space of N spins

Laplacian on Q_N: $(-\Delta\psi)(\sigma) := N\psi(\sigma) - \sum_{j=1}^N \psi(F_j\sigma)$

- Spin flip: $F_j\sigma = (\sigma_1, \dots, -\sigma_j, \dots, \sigma_N)$

  Hence the Laplacian acts as a **transversal magnetic field**: $-\Delta = N - \sum_{j=1}^N \sigma_j^x$

- Eigenvalues: $2|A|, \ \ A \subset \{1, \dots, N\}$ Degeneracies: $\binom{N}{|A|}$

  Normalized Eigenvectors: $f_A(\sigma) = \frac{1}{\sqrt{2}^N} \prod_{j\in A} \sigma_j$

Perturbation by a multiplication operator U:

$$\boxed{H = -\Delta + \kappa\, U}$$

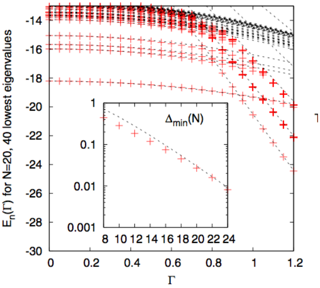
- $U = U(\sigma_1^z, \dots, \sigma_j^z)$; Coupling constant $\kappa \geq 0$; $\|U\|_\infty \approx \mathcal{O}(N)$
- **In this talk:** $\boxed{U(\sigma) = \sqrt{N}\, g(\sigma)}$ with $\{g(\sigma)\}_{\sigma\in Q_N}$ i.i.d. standard Gaussian r.v. **REM**

Predicted properties of the QREM

Predicted low-energy spectrum:

$\widehat{H} = \Gamma\,(-\Delta - N) + U/\sqrt{2}$,   i.e. $\kappa = (\sqrt{2}\,\Gamma)^{-1}$

Jörg/Krzakala/Kurchan/Maggs '08



First order phase transition of the ground state at $\kappa_c = \frac{1}{\sqrt{2\ln 2}}$:

$\kappa < \kappa_c$: Extended ground state with non-random ground-state energy

$$E_0 = -\kappa^2 + o(1)$$

$\kappa > \kappa_c$: Low lying eigenstates are concentrated on lowest values of U.

In particular: $E_0 = N + \kappa \min U + \mathcal{O}(1)$

$\kappa = \kappa_c$: Energy gap $\Delta_{min} = E_1 - E_0$ vanishes exponentially in N

Main aim in this talk: *explain some of the above features!*

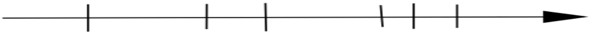
Some heuristics

Known properties of the **REM**: $U(\sigma) = \sqrt{N}\, g(\sigma)$

- Except for events of exponentially small probability:

$$U_0 := \min U = -\kappa_c^{-1}\, N + \mathcal{O}(\ln N)$$

- The extreme values $U_0 \leq U_1 \leq \ldots$ form a **Poisson process** about $-\kappa_c^{-1}\, N + \mathcal{O}(\ln N)$ with intensity $e^{-\kappa_c x} dx$.



Perturbation theory:

- Fate of localized states: $\langle \delta_\sigma , H \delta_\sigma \rangle = N + \kappa\, U(\sigma)$.
- Fate of delocalized states: $\langle f_A , U\, f_A \rangle = \frac{1}{2^N} \sum_\sigma U(\sigma) = \mathcal{O}(\sqrt{N}\, 2^{-N/2})$.

More motivations and related questions

1. Adiabatic Quantum Optimization: Farhi/Goldstone/Gutmann/Snipser '01, ...

Question: Find minimum in a complex energy landscape $U(\sigma)$

e.g. REM, Exact Cover 3, ...

Idea: Evolve the ground state through adiabatic quantum evolution, i.e. $i\,\partial_t \psi_t = H(t/\tau)\,\psi_t$ generated by

$$\boxed{H(s) := (1-s)(-\Delta) + s\,U\,, \qquad s \in [0,1]}$$

Required time : $\tau \approx c\,\Delta_{min}^{-2}$

2. Mean field model for localization transition in disordered N particle systems

Altshuler '06

3. Evolutionary Genetics: Rugged fitness landscape for quasispecies ...

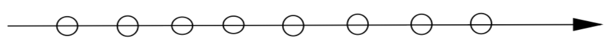
Schuster/Eigner '77, Baake/Wagner '01, ...

Theorem (Case $\kappa < \kappa_c$)

Except for events of exponentially small probability, the eigenvalues of H strictly below $\left(1 - \frac{\kappa}{\kappa_c} - \delta\right) N$ *are within balls centred at*

$$\boxed{2n - \frac{\kappa^2}{1 - \frac{2n}{N}}\,, \qquad n \in \mathbb{N}_0},$$

of radius $\mathcal{O}\left(N^{-\frac{1}{2}+\delta}\right)$ *with* $\delta > 0$ *arbitrary.*



There are exactly $\binom{N}{n}$ *eigenvalues in each ball and their eigenfunctions are delocalized:*

$$\boxed{\|\psi_E\|_\infty^2 \;\leq\; 2^{-N}\, e^{\Gamma\left(\frac{x_E}{2}\right)N}}$$

where $\Gamma(x) := -x \ln x - (1-x)\ln(1-x)$ *and* $x_E := \frac{E}{N} + \frac{\kappa}{\kappa_c} + \delta$.

Step 1: *Hypercontractivity of the Laplacian*

$$|\psi_E(\sigma)|^2 \leq \langle \delta_\sigma \,, P_{(-\infty,E]}(H)\, \delta_\sigma \rangle \;=\; \inf_{t>0}\, e^{tE} \langle \delta_\sigma \,, e^{-tH}\, \delta_\sigma \rangle$$
$$= \inf_{t>0}\, e^{t(E-\kappa U_0)} \langle \delta_\sigma \,, e^{t\Delta}\, \delta_\sigma \rangle \;=\; 2^{-N}\, e^{\Gamma\left(\frac{x_E}{2}\right)N} \,.$$

Step 2: *Reduction of fluctuations*

Illustration for the ground state energy $E_0(g) = \inf_{\|\psi\|_2=1} \langle \psi \,, H\,\psi \rangle$

Lipschitz continuity of as a function of the 2^N Gaussian random variables g

$$E_0(g') - E_0(g) \leq \kappa\sqrt{N} \sum_\sigma |g'(\sigma) - g(\sigma)|\, |\psi_0(\sigma)|^2$$
$$\leq \kappa\sqrt{N}\, \|\psi_0\|_4^2\, \|g - g'\|_2 \;\leq\; \kappa\sqrt{N}\, \|\psi_0\|_\infty\, \|g - g'\|_2$$
$$\leq \kappa\sqrt{N}\, 2^{-\frac{N}{2}}\, e^{\Gamma\left(\frac{x_0}{2}\right)\frac{N}{2}}\, \|g - g'\|_2 \,.$$

Hence E_0 is identically distributed to $\kappa\sqrt{N}\, 2^{-\frac{N}{2}}\, e^{\Gamma\left(\frac{x_0}{2}\right)\frac{N}{2}}$ times a 1-Lipschitz function of **one** normalized Gaussian.

The complete proof uses concentration of measure inequality à la Talagrand:

Lemma

There exist constants $C, c < \infty$ such that for any $\varepsilon > 0$ and any $\lambda > 0$:

$$\mathbb{P}\left(\left| \|P_\varepsilon U P_\varepsilon\| - \mathbb{E}\left[\|P_\varepsilon U P_\varepsilon\|\right] \right| > \lambda \sqrt{\frac{\dim P_\varepsilon}{2^N}} \right) \leq \; C\, e^{-c\lambda^2}$$

$$\mathbb{E}\left[\|P_\varepsilon U P_\varepsilon\|\right] \; \leq \; C\, N \sqrt{\frac{\dim P_\varepsilon}{2^N}}$$

where $P_\varepsilon := 1 - 1_{[N(1-\varepsilon), N(1+\varepsilon)]}(-\Delta)$ *and* $\varepsilon = N^{-\frac{1}{2}+\delta}$.

Step 3: *Schur complement formula*

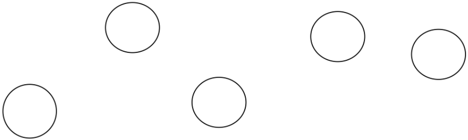
$$P_\varepsilon\,(H-z)^{-1}\,P_\varepsilon \;=\; \left(P_\varepsilon H P_\varepsilon - z - \kappa^2 P_\varepsilon U Q_\varepsilon\,(Q_\varepsilon H Q_\varepsilon - z)^{-1}\,Q_\varepsilon U P_\varepsilon \right)^{-1}$$

2. Low-energy regime of the QREM in case $\kappa \geq \kappa_c$

Main idea: *Geometric decomposition*

For energies below $E_\delta := \left(1 - \frac{\kappa}{\kappa_c} + \delta\right) N$ the localized eigenstates originate in **large negative deviation sites**:

$$\boxed{X_\delta := \left\{ \sigma \,|\, \kappa\, U(\sigma) < -\frac{\kappa}{\kappa_c} N + \delta N \right\}}$$



For $\delta > 0$ small enough and except for events of exponentially small probability (e.e.p.):

- X_δ consists of isolated points which are separated by a distance greater than $2\gamma N$ with some $\gamma > 0$.
- On balls $B_{\gamma,\sigma} := \{\sigma' \mid \text{dist}(\sigma, \sigma') < \gamma N\}$ the potential is larger than $-\epsilon N$ aside from at σ.

Low-energy regime of the QREM in case $\kappa \geq \kappa_c$

Theorem (Case $\kappa \geq \kappa_c$)

E.e.p. and for $\delta > 0$ sufficiently small, there is some $\gamma > 0$ such that all eigenvalues of H below $E_\delta = \left(1 - \frac{\kappa}{\kappa_c} + \delta\right) N$ coincide up to an exponentially small error with those of

$$\widehat{H}_\delta := H_R \oplus \bigoplus_{\sigma \in X_\delta} H_{B_{\gamma,\sigma}} \, .$$

where $R := Q_N \backslash \bigcup_{\sigma \in X_\delta} B_{\gamma,\sigma}$.

- Low energy spectrum of H_R looks like H in the delocalisation regime
- Low energy spectrum of $H_{B_{\gamma,\sigma}}$ is explicit ...

Known properties of Laplacian on $B_{\gamma,\sigma}$:

$$\boxed{E_0(-\Delta_{B_{\gamma,\sigma}}) = N(1 - 2\sqrt{\gamma(1-\gamma)}) + o(N)}$$

Adding a large negative potential κU at σ and some more moderate background elsewhere, rank-one analysis yields:

- $$\boxed{E_0(H_{B_{\gamma,\sigma}}) \;=\; N + \kappa U(\sigma) - s_\gamma(N + \kappa U(\sigma)) + \mathcal{O}(N^{-1/2})}$$
  where s_γ is the self-energy of the Laplacian on a ball of radius γN.
- for the corresponding normalised ground state:
  $$\sum_{\sigma' \in \partial B_{\gamma,\sigma}} \left|\psi_0(\sigma')\right|^2 \;\leq\; e^{-L_\gamma N} \qquad \text{for some } L_\gamma > 0.$$
  $$\left|\psi_0(\sigma)\right|^2 \geq 1 - \mathcal{O}(N^{-1})$$
- $H_{B_{\gamma,\sigma}}$ has a spectral gap of order N above the ground state.

1. Complete description of the **low-energy** spectrum of the QREM

   . . . and generalisations to non-gaussian r.v.'s

2. Ground-state phase transition at $\kappa = \kappa_c$ with an exponentially closing gap.

Low-energy spectrum:

$\widehat{H} = \Gamma\,(-\Delta - N) + U/\sqrt{2}$, i.e. $\kappa = (\sqrt{2}\,\Gamma)^{-1}$

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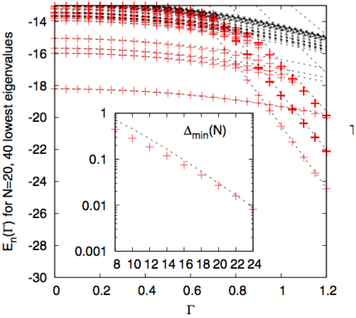


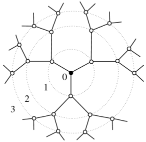
Existence of resonance delocalization for eigenvalues in gaps of Laplacian for intermediate energies

Project with M. Aizenman and M. Shamis (Princeton)

Eigenstates in the spectral gaps of the Laplacian will form delocalized states. They are however **not uniformely extented** *over the Hamming cube and are presumably another example of what physicists now call* **non-ergodic extended states**.

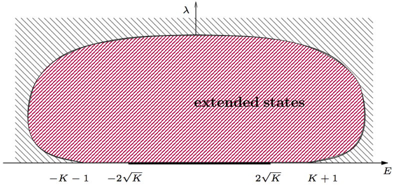
Inspired by: M.A., S.W.: *Resonant delocalization for random Schrödinger operators on tree graphs, JEMS* (2013)

Regular tree graph $\mathbb{B}$ with coordination number $K+1 \geq 3$:



$$\boxed{H = -\Delta + \lambda\, g \qquad \text{on } \ell^2(\mathbb{B}).}$$

Spectral phase diagram:



Appearance of extended states within the ℓ^1-spectrum of $-\Delta$