Quantum Hall phases and plasma analogy in rotating Bose gases

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References

N. Rougerie, S. Serfaty, J.Y., *Quantum Hall states of bosons in rotating anharmonic traps, Phys. Rev. A* **87**, 023618 (2013), arXiv:1212.1085

N. Rougerie, S. Serfaty, J.Y., *Quantum Hall Phases and the Plasma Analogy in Rotating Trapped Bose Gases, J. Stat. Phys*, (DOI) 10.1007/s10955-013-0766-0 (Special issue for Herbert Spohn), arXiv:1301.1043

FQHE

- One of the most striking phenomena in condensed matter physics is the Fractional Quantum Hall Effect (FQHE) for charged fermions in strong magnetic fields that still, after decades of research, poses many unresolved questions.
- It has been recognized for some time that bosonic analogues of the FQHE can be studied in cold quantum gases set in rapid rotation in harmonic traps: There is a transition from an uncorrelated Hartree states to strongly correlated many-body states such as as the Laughlin wave function when the rotational velocity approaches the frequency of the trap.
- I shall focus on *one* aspect of the Quantum Hall Physics of cold bosons: The emergence of strongly correlated many-body states in anharmonic traps through appropriate tuning of the parameters and changes in the properties of these states as the parameters are varied.

Hamiltonian

The Hamiltonian for N spinless bosons in a rotating frame with angular velocity $\mathbf{\Omega} = \Omega \mathbf{e}_3$ is

$$H^{3\mathrm{D}} = \sum_{j=1}^{N} \left(-\tfrac{1}{2}\Delta_j + V(\mathbf{x}_j) - \mathbf{L}_j \cdot \mathbf{\Omega} \right) + \sum_{1 \leq i < j \leq N} v(|\mathbf{x}_i - \mathbf{x}_j|).$$

Here $\mathbf{x}_j \in \mathbb{R}^3$, $\mathbf{L} = -\mathrm{i}\,\mathbf{x} \wedge \nabla$ is the angular momentum operator, V is a confining external potential and v the interaction potential. Units are chosen so that $\hbar = m = 1$.

We are interested in ground state properties of $H_N^{3\mathrm{D}}$ for large N.

Hamiltonian (cont.)

An interesting situation occurs when V is a quadratic potential in the direction $\perp$ to the rotation axis

$$V(\mathbf{x}) = \tfrac{1}{2}\Omega_\perp^2 r^2 + V^{\|}(x_3)$$

with $r^2 = \mathbf{x}_1^2 + x_2^2$ and the angular velocity Ω approaches the frequency of the potential $\Omega_\perp$ from below.

It is useful to write the Hamiltonian in a 'magnetic' form:

$$H^{3\mathrm{D}} = \sum_{j=1}^{N} \left\{ \tfrac{1}{2}(\mathrm{i}\nabla_j + \mathbf{A}(\mathbf{x}_j))^2 + \omega\, \mathbf{e}_3 \cdot \mathbf{L}_j + V^{\|}(x_3) \right\} + \sum_{i<j} v(|\mathbf{x}_i - \mathbf{x}_j|)$$

where $\mathbf{A}(\mathbf{x}_j) = \Omega_\perp(x_2, -x_1, 0)$ and

$$\omega = \Omega_\perp - \Omega > 0$$

LLL and Bargmann Space

For $\omega \ll \min\{\Omega_\perp, \text{gap in } x_3\text{-direction}\}$ the ground state state becomes effectively 2D and restricted to the Lowest Landau Level (LLL) of the magnetic Laplacian with energy zero (after subtraction). We choose units so that $\Omega_\perp = 1$.

The relevant Hilbert space is the Bargmann space $\mathcal{B}_N$ of symmetric, analytic functions ϕ of $z_1, \dots, z_N \in \mathbb{C}$ such that

$$\int_{\mathbb{C}^N} |\phi(z_1, \dots, z_N)|^2 \exp\left(-\sum_{j=1}^N |z_j|^2\right) \mathrm{d}^2 z_1 \cdots \mathrm{d}^2 z_N < \infty.$$

The 'free' part of the Hamiltonian in the LLL is (up to a constant)

$$\omega \mathcal{L}_N$$

with the angular momentum operator

$$\mathcal{L}_N = \sum_{i=1}^N z_i \partial_{z_i}.$$

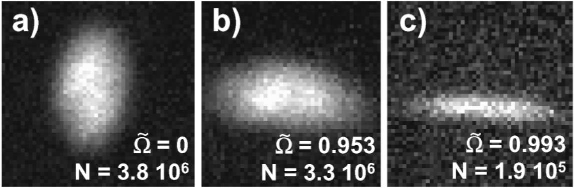


FIG. 1. Side view images of BECs in trap. (a) Static BEC. The aspect ratio $R_z/R_\rho = 1.57$ ($N = 3.8 \times 10^6$ atoms) resembles the prolate trap shape. (b) After evaporative spin-up, $N = 3.3 \times 10^6$, $\tilde{\Omega} = 0.953$ and (c) evaporative plus optical spin-up, $N = 1.9 \times 10^5$, $\tilde{\Omega} = 0.993$. Because of centrifugal distortion the aspect ratio is changed by a factor of 8 compared to (a).

Contact interaction

For short range, nonnegative interaction potentials v Lewin and Seiringer (2009) have shown that for $\omega a \ll 1$, with a the scattering length of v, the motion is, indeed, restricted to the 2D LLL, and moreover that $v(\mathbf{x}_i - \mathbf{x}_j)$ can be replaced by $g\delta(z_i - z_j)$ with

$$g \sim a\Omega_{\|}^{1/2} > 0$$

Such a potential is perfectly acceptable for analytic functions and is even given by a bounded operator on the Bargmann space:

Define δ_{12} on $\mathcal{B}_2$ by

$$\delta_{12}\phi(z_1, z_2) = \frac{1}{(2\pi)^{3/2}}\phi\big(\tfrac{1}{2}(z_1 + z_2), \tfrac{1}{2}(z_1 + z_2)\big).$$

Then a computation using the analyticity of ϕ shows that

$$\langle \phi, \delta_{12} \rangle = \int_{\mathbb{C}} |\phi(z, z)|^2 \exp(-2|z|^2)\, \mathrm{d}^2 z.$$

Yrast spectrum

The Hamiltonian on Bargmann space is now (apart from a constant)

$$H_N = \omega\, \mathcal{L}_N + g\, \mathcal{I}_N$$

with

$$\mathcal{L}_N = \sum_{i=1}^{N} z_i \partial_i \qquad \mathcal{I}_N = \sum_{i<j} \delta_{ij}$$

Notable feature: The operators $\mathcal{L}_N$ and $\mathcal{I}_N$ commute. The lower boundary of (the convex hull of) their joint spectrum is called the Yrast curve.

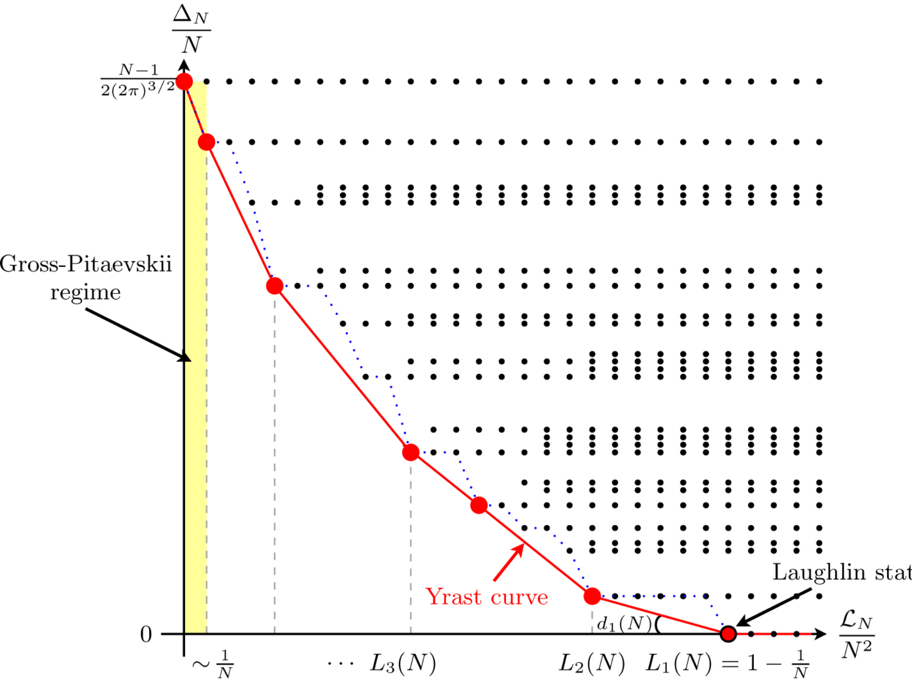


FIGURE 1. *General form of the joint spectrum of* Δ_N/N *and* $\mathcal{L}_N/N^2$. *The dashed curve is the graph of* $\ell \mapsto \Delta_N(\ell N^2)/N$, *whereas the solid one is the* yrast curve. *Points of the joint spectrum lying on the yrast curve are emphasized by thick dots. This figure represents only a sketch, numerical studies for the joint spectrum of* Δ_N *and* $\mathcal{L}_N$ *can be found in* [40, 35, 6].

Gaps

For every value of the angular momentum L the interaction operator $\mathcal{I}_N$ has a nonzero spectral gap

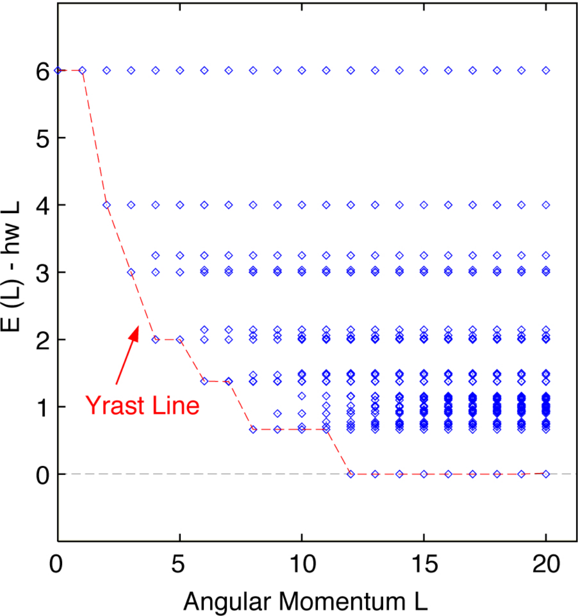
$$\Delta(L) = \inf\{\mathrm{spec}\, \mathcal{I}_N \restriction_{\mathcal{L}_N = L} \setminus \{0\}\} > 0$$

The gap, and hence the Yrast curve, are *monotonously decreasing* with L. Reason: the angular momentum of an eigenstate of $\mathcal{I}_N$ can be increased by one unit by multiplying with the center of mass coordinate $(z_1 + \cdots + z_N)/N$.

There is numerical and some theoretical evidence that

$$\Delta(L) \geq \Delta(N(N-1) - N) \geq C \equiv \Delta > 0$$

for all L independently of N but this is still not proved.



E (L) - hw L
Angular Momentum L
Yrast Line
0
1
2
3
4
5
6
0
5
10
15
20

Passage to the Laughlin state

For a given ratio ω/g the angular momentum and interaction energy of the ground state(s) ψ_0 is determined by the point(s) on the Yrast curve where a supporting line has slope $-\omega/g$.

For $|\omega|/g \gg 1/N$, ψ_0 is an uncorrelated Hartree state for large N (Lieb, Seiringer, JY, 2009)

If $|\omega|/g < \Delta/N^2$ one reaches the Laughlin state, whose wave function in Bargman space is

$$\psi_{\text{Laughlin}}(z_1, \ldots, z_N) = c \prod_{i<j} (z_i - z_j)^2.$$

It has interaction energy 0 and angular momentum $N(N-1)$.

The limit $\omega \to 0$ keeping $\omega > 0$ is experimentally very delicate, however.

Anharmonic potential

For stability but also to study new effects we consider now a modification of the Hamiltonian:

$$H^{3\mathrm{D}} \to H^{3\mathrm{D}} + k \sum_{i=1}^{N} |z_i|^4$$

with a new parameter $k > 0$. The potential $|z|^4$ can be expressed through $\mathcal{L}$ and $\mathcal{L}^2$ on Bargmann space and the Hamiltonian can be written (up to an additive constant)

$$H_N = (\omega + 3k)\mathcal{L}_N + k \sum_{i=1}^{N} \mathcal{L}_{(i)}^2 + g\,\mathcal{I}_N$$

Powers of angular momentum are equivalent to powers of $|z|^2$.

With $\mathcal{L} = z\partial$ on the Bargmann space $\mathcal{B}$ we have by partial integration, using the analyticity of φ,

$$\langle \varphi, \mathcal{L}\varphi \rangle = \int |\varphi(z)|^2 (|z|^2 - 1) \exp(-|z|^2) \, \mathrm{d}^2 z$$

and

$$\langle \varphi, \mathcal{L}^2\varphi \rangle = \int (|z|^4 - 3|z|^2 + 1) |\varphi(z)|^2 \exp(-|z|^2) \, \mathrm{d}^2 z$$

Anharmonic potential (cont.)

Another viewpoint: Consider the energy as a functional on the LLL Hilbert space $\subset L^2(\mathbb{C})$ consisting of wave functions of the form $\Psi = \psi(z_1, \dots, z_N)\exp(-\sum_j |z_j|^2/2)$, $\psi \in \mathcal{B}_N$:

$$\mathcal{E}[\Psi] = \int V_{\omega,k}(z)\rho_\Psi(z) + \langle \Psi, \mathcal{I}_N \Psi \rangle$$

where ρ_Ψ is the one-particle density of Ψ with the normalization $\int \rho_\Psi(z)\,\mathrm{d}^2 z = N$ and the potential is

$$V_{\omega,k}(z) = \omega\,|z|^2 + k\,|z|^4.$$

Note that $\omega < 0$ is allowed provided $k > 0$.

Fully correlated states

We shall call states $\Psi \in \ker \mathcal{I}_N$ fully correlated. These are states of the form

$$\Psi(z_1, ...z_N) = \phi(z_1, \dots, z_N)\Psi_{\text{Laugh}}(z_1, ...z_N)$$

with ϕ symmetric and analytic, and the Laughlin state

$$\Psi_{\text{Laugh}}(z_1, ...z_N) = c \prod_{i<j} (z_i - z_j)^2 \, e^{-\sum_{j=1}^N |z_j|^2/2}.$$

For the Hamiltonian *without* the anharmonic addition to the potential the Laughlin state is an *exact* ground state This is *not* true for $k \neq 0$ because $\sum_{i=1}^N \mathcal{L}_{(i)}^2$ does not commute with $\mathcal{I}_N$.

Note, however, that $\mathcal{L}_N$ still commutes with the Hamiltonian.

Criterion for strong correlations in the ground state Ψ_o

THEOREM 1.

$$\left\| P_{\mathrm{Ker}(\mathcal{I}_N)^\perp} \Psi_0 \right\| \to 0$$

in the limit $N \to \infty$, $\omega, k \to 0$ if one of the following conditions hold:

- $\omega \geq 0$ and $\omega N^2 + kN^3 \ll g\,\Delta$.
- $0 \geq \omega \geq -2kN$ and $N(\omega^2/k) + \omega N^2 + kN^3 \ll g\,\Delta$.
- $\omega \leq -2kN$, $|\omega|/k \lesssim N^2$ and $kN^3 \ll g\,\Delta$
- $\omega \leq -2kN$, $|\omega|/k \gg N^2$ and $|\omega|N \ll g\,\Delta$

Note: For $k = 0$ the first item is just the condition for the passage to the Laughlin state, while the other conditions are void because $\omega < 0$ is only allowed if $k > 0$.

Trial States

Apart from a simple lower bound, the essential ingredient of the proof of the theorem is an upper bound to the energy of trial states of the form 'giant vortex times Laughlin', namely, with $m \geq 0$ and $c_{m,N}$ a normalization constant,

$$\Psi_{\rm gv}^{(m)}(z_1, \dots, z_N) = c_{m,N} \prod_{j=1}^{N} z_j^m \prod_{i<j} (z_i - z_j)^2 \, e^{-\sum_{j=1}^N |z_j|^2/2}$$

For small m these are Laughlin's 'quasi hole' states. For $m \gtrsim N$, i.e., $mN \gtrsim N^2$= angular momentum of the Laughlin state, there label 'giant vortex' appears more appropriate.

Optimal Choice of m

The energy of the trial states can be estimated using properties of the angular momentum operators and the radial symmetry in each variable of $\prod_{j=1}^{N} |z_j|^{2m} \times$ the gaussian measure. Optimizing the estimate over m leads to

$$m_{\text{opt}} = \begin{cases} 0 & \text{if } \omega \geq -2kN \\ \frac{|\omega|}{2k} - N & \text{if } \omega < -2kN. \end{cases}$$

This is consistent with the picture that the Laughlin state is an approximate ground state in the first two cases of Theorem 1, in particular for negative ω as long as $|\omega|/k \lesssim N$. The angular momentum remains $O(N^2)$ in these cases.

Transitions at $|\omega|/k \sim N$ and at $|\omega|/k \sim N^2$

When $\omega < 0$ and $|\omega|/(kN)$ becomes large the angular momentum is approximately $L_{\mathrm{qh}} = O(N|\omega|/k) \gg N^2$, much larger than for the Laughlin state.

A further transition at $|\omega|/k \sim N^2$ is manifest through the change of the subleading contribution to the energy of the trial functions. Its order of magnitude changes from $O(kN^3)$ to $O(|\omega|N)$ at the transition.

To obtain further insights into the physics of the transition we consider the density of the trial wave functions.

The N-particle density as a Gibbs measure

We denote $(z_1, \dots, z_N)$ by Z for short and consider the scaled N particle density (normalized to 1)

$$\mu_{N,m}(Z) := N^N \left| \Psi_{\mathrm{gv}}^{(m)}(\sqrt{N} Z) \right|^2 .$$

We can write

$$\begin{aligned} \mu_{N,m}(Z) &= \mathcal{Z}_{N,m}^{-1} \exp\left(\sum_{j=1}^{N} \left(-N|z_j|^2 + 2m \log|z_j| \right) - 4 \sum_{i<j} \log|z_i - z_j| \right) \\ &= \mathcal{Z}_{N,m}^{-1} \exp\left(-\frac{1}{T} \mathcal{H}_{N,m}(Z) \right), \end{aligned}$$

with $T = N^{-1}$ and

$$\mathcal{H}_{N,m}(Z) = \sum_{j=1}^{N} \left(-|z_j|^2 + \frac{2m}{N} \log|z_j| \right) - \frac{4}{N} \sum_{i<j} \log|z_i - z_j|.$$

Plasma analogy and mean field limit

The Hamiltonian $\mathcal{H}_{N,m}(Z)$ defines a classical 2D Coulomb gas ('plasma') in a uniform background of opposite charge and a point charge $(2m/N)$ at the origin, corresponding respectively to the $-|z_i|^2$ and the $\frac{2m}{N}\log|z_j|$ terms.

The probability measure $\mu_{N,m}(Z)$ minimizes the free energy functional

$$\mathcal{F}(\mu) = \int \mathcal{H}_{N,m}(Z)\mu(Z) + T\int \mu(Z)\log\mu(Z)$$

for this Hamiltonian at $T = N^{-1}$.

The $N \to \infty$ limit is in this interpretation a mean field limit where at the same time $T \to 0$. It is thus not unreasonable to expect that for large N,

$$\mu_{N,m} \approx \rho^{\otimes N}$$

with a one-particle density ρ minimizing a mean field free energy functional.

Mean field limit (cont.)

The mean field free energy functional is defined as

$$\mathcal{E}^{\mathrm{MF}}_{N,m}[\rho] := \int_{\mathbb{R}^2} W_m\,\rho + 2\int\int \rho(z)\log|z-z'|\rho(z') + N^{-1}\int_{\mathbb{R}^2}\rho\log\rho$$

with

$$W_m(z) = |z|^2 - 2\frac{m}{N}\log|z|.$$

It has a minimizer $\rho^{\mathrm{MF}}_{N,m}$ among probability measures on $\mathbb{R}^2$ and this minimizer should be a good approximation for the scaled 1-particle density of the trial wave function, i.e.,

$$\mu^{(1)}_{N,m}(z) := \int_{\mathbb{R}^{2(N-1)}} \mu_{N,m}(z, z_2, \dots, z_N)\mathrm{d}^2 z_2 \dots \mathrm{d}^2 z_N.$$

The Mean Field Limit Theorem

H. Spohn and M. Kiessling have previously studied such mean field limits, using compactness arguments. For our purpose, however, we need quantitative estimates on the approximation of $\mu^{(1)}_{N,m}$ by $\rho^{\mathrm{MF}}_{N,m}$.

THEOREM 2

There exists a constant $C > 0$ such that for large enough N and any $V \in H^1(\mathbb{R}^2) \cap W^{2,\infty}(\mathbb{R}^2)$

$$\left| \int_{\mathbb{R}^2} \left(\mu^{(1)}_{N,m} - \rho^{\mathrm{MF}}_{N,m} \right) V \right| \leq C (\log N/N)^{1/2} \|\nabla V\|_{L^1} + C N^{-1} \|\nabla^2 V\|_{L^\infty}$$

if $m \lesssim N^2$, and

$$\left| \int_{\mathbb{R}^2} \left(\mu^{(1)}_{N,m} - \rho^{\mathrm{MF}}_{N,m} \right) V \right| \leq C N^{-1/2} m^{-1/4} \|V\|_{L_\infty}$$

for $m \gg N^2$.

Ingredients of the Proof of Theorem 2

The proof of Theorem 2 is based on upper and lower bounds for the free energy.

For the upper bound one uses $\rho^{\mathrm{MF}\otimes N}$ as a trial measure. The lower bound uses:

- 2D versions of two classical electrostatic results: Onsager's lemma, and an estimate of the change in electrostatic energy when charges are smeared out.
- The variational equation associated with the minimization of the mean field free energy functional.
- Positivity of relative entropies, more precisely the Cszizàr-Kullback-Pinsker inequality

The estimate on the density (for $m \lesssim N^2$) follows essentially from the fact that the positive Coulomb energy $D(\mu^{(1)} - \rho^{\mathrm{MF}}, \mu^{(1)} - \rho^{\mathrm{MF}})$ is squeezed between the upper and lower bounds to the free energy.

Use of Theorem 2

The use of Theorem 2 is threefold:

- It provides a picture of the one-particle density of the trial states.
- Approximating the energy of the trial functions by

$$\int V_{\omega,k}(z)\rho^{\mathrm{MF}}_{N,m}(z)\mathrm{d}^2r$$

  with an optimal m improves the previous energy estimates computed by angular momentum considerations.
- Estimating the energy with the aid of the angular momentum operators requires potentials of a special form. The 'density method' is more general.

Asymptotic Formulas for the Mean Field Density

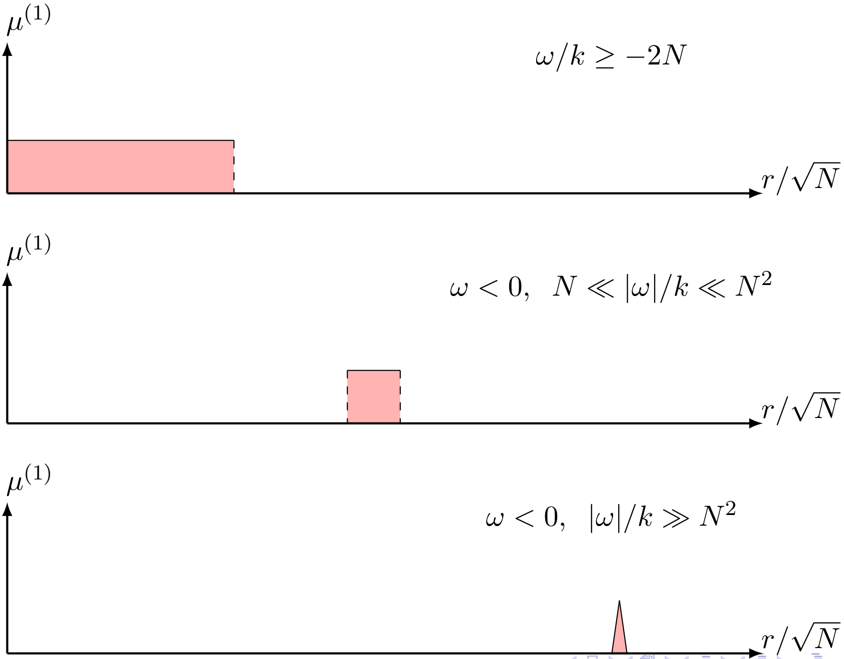
The picture of the 1-particle density arises from asymptotic formulas for the mean-field density: If $m \leq N^2$, then ρ_m^{MF} is well approximated by a density $\hat{\rho}_m^{\mathrm{MF}}$ that minimizes the mean field functional without the entropy term. It takes a constant value $(2\pi)^{-1}$ on an annulus with inner and outer radia $R_- = (m/N)^{1/2}$ and $R_+ = (2 + m/N)^{1/2}$ and is zero otherwise. The constant value is a manifestation of the incompressibility of the density of the trial state.

For $m \gtrsim N^2$ the entropy term dominates the interaction term $\int\int \rho(z) \log |z - z'| \rho(z')$. The density is well approximated by the gaussian $\rho^{\mathrm{th}}(z) = (\pi m!)^- 1 |z|^{2m} \exp(-|z|^2)$ that is centered around $\sqrt{m}$.

Phases

As the parameters ω and k tend to zero and N is large the qualitative properties of the optimal trial wave functions thus exhibit different phases:

- The state changes from a pure Laughlin state to a modified Laughlin state with a 'hole' in the density around the center when ω is negative and $|\omega|$ exceeds $2kN$.
- A further transition is indicated at $|\omega| \sim kN^2$. The density profile changes from being 'flat' to a Gaussian.


$\mu^{(1)}$
$\omega/k \geq -2N$
$r/\sqrt{N}$
$\mu^{(1)}$
$\omega < 0, \quad N \ll |\omega|/k \ll N^2$
$r/\sqrt{N}$
$\mu^{(1)}$
$\omega < 0, \quad |\omega|/k \gg N^2$
$r/\sqrt{N}$

Summary and conclusions

- We have studied a rotating Bose gas in a quadratic plus quartic trap where the rotational frequency can exceed the frequency of the quadratic part of the trap.
- Through the analysis of trial states for energy upper bounds and simple lower bounds we have obtained criteria for the ground state to be fully correlated in an asymptotic limit. The lower bounds, although not sharp, are of the same order of magnitude as the upper bounds.
- The density of the wave functions can be analyzed through the plasma analogy. The character of the density changes at $|\omega|/k = O(N)$ and again at $|\omega|/k = O(N^2)$.
- Sharp lower bounds, that would follow from a general proof of incompressibility of fully correlated states, would establish the observed crossing of optimal trial functions as a genuine quantum phase transition.