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Kinetic Limit for the Hubbard Model

joint work with

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classical particles at low density

⇝ Boltzmann equation

Lanford (1975)

Corresponding program for

wave turbulence

- weakly nonlinear wave equations
- weakly interacting quantum fluids

work horse of solid state physics

HERE: lattice fermions

1. Interacting lattice fermions

Hubbard model

- spin $\frac{1}{2}$ fermions on $\mathbb{Z}^d$

$$H = \sum_{\substack{x,y \\ \sigma=\pm 1}} \alpha(x-y)\, a(x,\sigma)^* a(y,\sigma)$$

$$+ \lambda \sum_x a(x,1)^* a(x,1)\, a(x,-1)^* a(x,-1)$$

on-site interaction

hopping $\alpha \in \mathbb{R}$

$\alpha(x) = \alpha(-x)$

finite range

DISPERSION $\omega(k) = \hat{\alpha}(k)$

$$N_\sigma = \sum_x a(x,\sigma)^* a(x,\sigma)$$

are conserved

$U(2)$ symmetry

spinless fermions

$$H = \sum_{x,y} \alpha(x-y)\, a(x)^* a(y) \; + \lambda \sum_{x,y} V(x-y)\, a(x)^* a(y)^* a(y)\, a(x)$$

potential $V(x) = V(-x)$
finite range

|| on site is trivial ||

$\Rightarrow$ renormalized dispersion

$$\omega^\lambda(k) = \omega(k) + \lambda R_\lambda(k)$$

$$R_\lambda(k_1) = \int dk_2\, W_\lambda(k_2)\left(\hat V(0) - \hat V(k_1-k_2)\right)$$

Hubbard: $R_\lambda(k)$ independent of k ✓

spinless: ?

| lattice bosons
| ok, up to equilibrium estimate

2. Kinetic limit

- spatially homogeneous, $\mathbb{Z}^d$

$\lambda \to 0$, $t \to \infty$, $\lambda^2 t$ fixed

spatial scale λ^{-2}

$t = 0$ $\quad \langle \cdot \rangle$ quasi free

$$\langle a \rangle = 0 \qquad \langle a a \rangle = 0$$

$$\langle a(k)^* a(k') \rangle = \delta(k-k')\, W(k)$$

$t > 0$ $\quad \langle a(k,t)^* a(k',t) \rangle = \delta(k-k')\, W_\lambda(k,t)$

PROVE:

$$\lim_{\lambda \to 0} W_\lambda(k, \lambda^{-2} t) = W(k,t)$$

// $W(k,t)$ solves Nordheim-Boltzmann equation //

$$W(k,0) = W(k)$$

Hugenholtz 1983

Ho, Landau 1997

// second order term

$$W(k) + \lambda^2 t\, \mathcal{C}(W)(k)$$

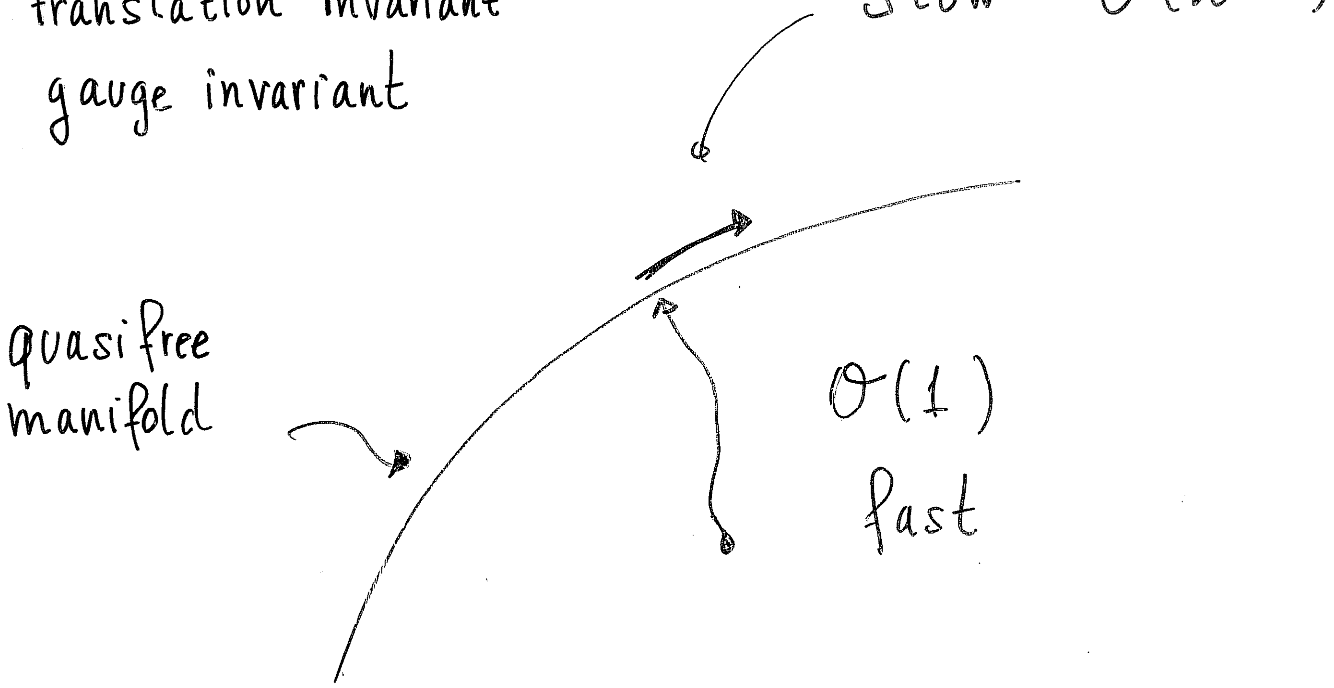
+ convergence to quasi free state under H_0 dynamics

$\rightsquigarrow$ kinetic picture

space of states

translation invariant
gauge invariant

slow $\mathcal{O}(\lambda^{-2})$



Nordheim - Boltzmann (1928)

H - theorem

$$\frac{\partial}{\partial t} W(k,t) = \mathcal{C}(W(t))(k)$$

$$\mathcal{C} W(k_1) = \pi \int_{(\mathbb{T}^d)^3} dk_2 dk_3 dk_4 \, \delta(k_1+k_2-k_3-k_4)\, \delta(\omega_1+\omega_2-\omega_3-\omega_4)$$

momentum energy

$$|\hat{V}(k_2-k_3)-\hat{V}(k_2-k_4)|^2 \left[\tilde{W}_1 \tilde{W}_2 W_3 W_4 - W_1 W_2 \tilde{W}_3 \tilde{W}_4 \right]$$

$$\tilde{W} = 1 - W$$

- cubic
- global solution Dolbeaut (1994)

$$0 \le W(k,t) \le 1$$

- collisional invariants $\rightsquigarrow$

$d \geq 2$, only stationary solutions are

$$W_{\beta,\mu} = \left(1 + e^{+\beta(\omega(k)-\mu)}\right)^{-1}$$

H.S. (2006)

expansion in λ

n-th order

classical $(c)^n$

$$\underbrace{\lambda^n \frac{1}{n!} t^n}_{\text{time ordering}} \quad \underbrace{n!}_{\text{nonlinearity}} \quad \underbrace{n!}_{\text{initial state}}$$

cut series at $\lambda n! = 1$

→ ERROR term ←

// proof of kinetic limit remains OPEN //

Benedetto, Castella, Esposito, Pulvirenti 2003 - 2008

- particles in $\mathbb{R}^d$, $d \geq 3$
- term-by-term convergence (all orders)
- Boltzmann statistics

our result

Hubbard model, $d \geq 4$, α = lattice Laplacian

- finite kinetic time

// time correlations in equilibrium //

by-product

term-by-term convergence

(any kinetic time)

3. Equilibrium time correlations

$\langle \cdot \rangle$ $\quad \beta$ KMS state , $|\lambda| \ll 1$.

chemical potential μ , $\mu < \min_k \omega(k)$

- ℓ_1 - clustering Salmhofer (2008)

$n \geq 4$ fully truncated

$$\sum_{\underline{x} \in \mathbb{Z}^{nd}} \delta_{x_1 0} \left| \langle \prod_{j=1}^{n} a(x_j)^{\#} \rangle^T \right| \leq \underset{\sim}{\lambda} \, (c_0)^n$$

NLS bosons $n!$

Hubbard

$$\langle a(x,\sigma,t)^* a(y,\sigma') \rangle = \int dk \, e^{ik\cdot(y-x)} \, C_\sigma^\lambda(k,t) \, \delta_{\sigma\sigma'}$$

RESULT

$$\lim_{\lambda \to 0} e^{-it(\omega(k)+\lambda R)/\lambda^2} C_\sigma^\lambda(k, \lambda^{-2} t) = C_\sigma(k,t)$$

$C_\sigma(k,t)$ damped oscillations // explicit formulae // $\quad \Big\| \quad$ $d \geq 4$, $0 \leq t \leq t_0$

- multipoint

$$< \prod_{j=1}^{n} a(x_j, t)^{\#} > \qquad \text{ok}$$

fixed lattice point

- number / energy current like fluctuations

$$\xi_\Lambda = |\Lambda|^{-1/2} \sum_{x,y \in \Lambda} \eta(x-y)\left(a(x)^* a(y) - \langle a(x)^* a(y) \rangle\right)$$

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} \langle\langle \xi_\Lambda(0)\, \xi_\Lambda(t) \rangle\rangle$$

Kubo inner product

$$\langle\langle A, B \rangle\rangle = \frac{1}{\beta} \int_0^\beta d\lambda \left(< A^* e^{-\lambda H} B e^{\lambda H} > - < A^* > < B > \right)$$

limit governed by

linearized Nordheim-Boltzmann

TO BE DONE

4. Dynamic input

A) ℓ_3 - bound

$$p_t(x) = \int_{\mathbb{T}^d} e^{-i\omega(k)t} e^{ik\cdot x} \, dk$$

$$\sum_x |p_t(x)|^3 \le c(1+|t|)^{-(1+\delta)} \qquad \delta > 0$$

$d \geq 3$, stationary phase

B) constructive interference

"Bad" set $\mathcal{M}$

$$\left| \int_{\mathbb{T}^d} e^{-it(\omega(k) \pm \omega(k-k_0))} dk \right| \le \frac{c}{1+|t|} \frac{1}{d(k_0, \mathcal{M})}$$

distance

- $\mathcal{M}$ is a finite collection of smooth curves

requires $d \geq 4$

C) crossing bounds

Erdös, Yau (2001)

Chen (2005)

Lukkarinen (2007)

in case of random Schrödinger

$d \geq 3$

5. Duhamel expansion

$\Rightarrow$ error bound

- interaction picture

$$\frac{d}{dt} \prod_{j=1}^{n} a(k_j, t)^{\#}$$

$$= \sum_{\ell=1}^{n} \Big(\prod_{j=1}^{\ell-1} a(k_j,t)^{\#} \Big) \frac{d}{dt} a(k_\ell, t) \Big(\prod_{j=\ell+1}^{n} a(k_j,t)^{\#} \Big)$$

operator ordering

- iterate

monomial order n

$$\hat a(k,t)^{\#} = \sum_{n=0}^{N-1} \mathcal{F}_n(k;t)\,[\hat a(0)^{\#}]$$

order n

$$\underbrace{+ \int_0^t ds\, \mathcal{F}_N(t-s,k)\,[\hat a(s)^{\#}]}_{\text{error term}}$$

full time evolution

- cut at $N! = \lambda$
- average over KMS state

$\Rightarrow$ oscillating Feynman diagrams

ERROR

$$|\langle a_{error}(t)^* \, a(0)\rangle|^2$$

$$= |\langle \int_0^t ds\, \mathcal{F}_N(t-s)\, [a(s)^\#]\, a(0)\rangle|^2$$

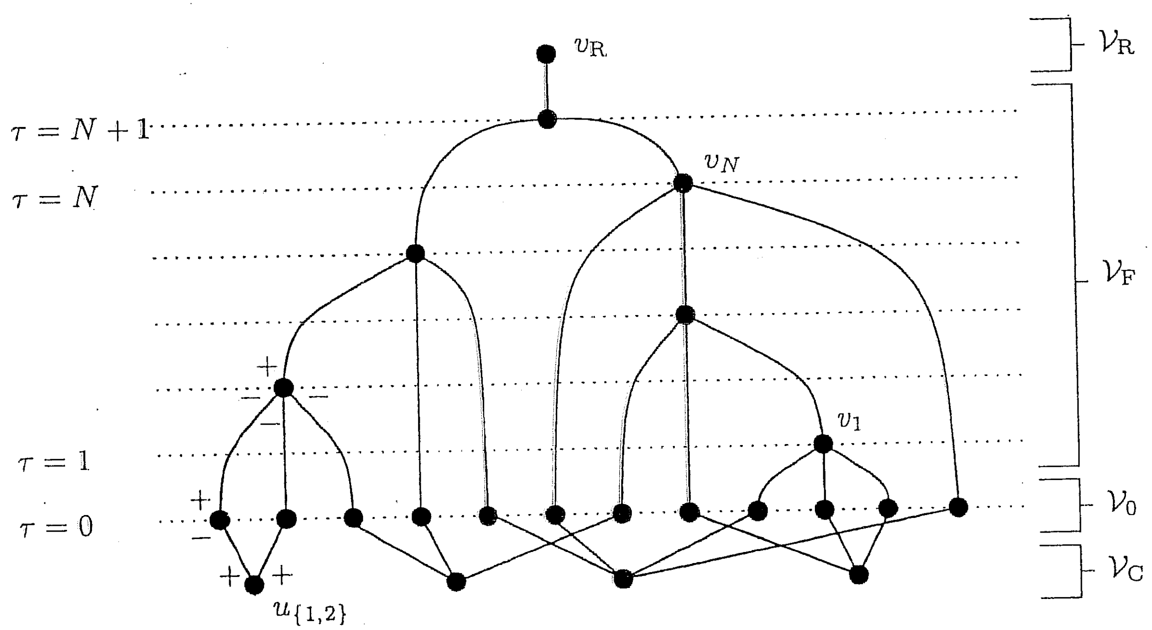
Schwarz

$$\leq \underbrace{t}_{O(\lambda^{-2})} \underbrace{\langle a(0)^* a(0)\rangle}_{\text{bounded}} \int_0^t ds\, \mathcal{F}_N(t-s)^* \mathcal{F}_N(t-s)\, \langle [a(s)^\#]^* [a(s)^\#]\rangle$$

stationarity

similar to Duhamel expansion of nonequilibrium initial state

main term converges only for kinetic time $|t| < t_0$!



v_{R}
v_N
v_1
$u_{\{1,2\}}$
$\tau = N+1$
$\tau = N$
$\tau = 1$
$\tau = 0$
$\mathcal{V}_{\mathrm{R}}$
$\mathcal{V}_{\mathrm{F}}$
$\mathcal{V}_0$
$\mathcal{V}_{\mathrm{C}}$

Conclusions

- $\langle a \rangle \neq 0$

  Bose - Einstein condensation

- Can one devise non-perturbative techniques ?

  candidate: nonlinear Schrödinger equation